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THE

DOCTRINE

OF

ULTIMATORS.

CONTAINING A

NEW ACQUISITION

TO

Mathematical Literature,

Naturally reſulting from the Conſideration of an Equation, as reducible from its *variable* to its *ultimate State*:

Or, a DISCOVERY

Of the true and genuine Foundation of what has hitherto miſtakenly prevailed under the improper Names of *Fluxions* and the *Differential Calculus*.

BY MEANS OF WHICH

We now have that Apex of all Mathematical Science entirely reſcued from the blind and ungeometrical Method of Deduction, which it has hitherto laboured under; and made to depend upon Principles as ſtrictly demonſtrable, as the moſt ſelf-evident Propoſition in the firſt Elements of *Geometry*.

By the Reverend Mr. JOHN KIRKBY,
Vicar of WALDERSHARE *in* KENT.

LONDON:
Printed for J. HODGES, at the *Looking-Glaſs* over-againſt St. *Magnus*'s Church, *London-Bridge*. MDCCXLVIII.

TO

My Worthy *BENEFACTORS*,

Sir EDWARD HALES,

Of St. *Stephens*, in the County of *Kent*, *Baronet*.

Humphry Pudner, and *John Sympſon*, Eſqs.

Of the City of *Canterbury*.

The following Sheets are humbly inſcrib'd:

AS

A grateful Acknowledgment, from

Their Moſt

London
May 25, 1748.

Obedient Servant,

JOHN KIRKBY.

[illegible]

THE PREFACE.

THE following Piece is an Extract, from a deſignedly much larger Work, containing only ſuch Mathematical Diſcoveries, as I have good Reaſon to believe can no where elſe be met with. And as ſuch I hope they will not be unacceptable to the Learned.

The Inveſtigation of the univerſal Theorems in the ſecond and fourth Sections, I take to be Diſcoveries entirely new. And their Application (in the fifth and ſixth Sections) for reverſing an infinite Series, and meaſuring the Arch of a Circle, is a ſufficient Proof of their Excellence above any Invention, that I know of, for the ſame Purpoſe.

What follows relates wholly to the Doctrine of Ultimators, *which depends upon ſcarce any Thing elſe but a due Application of the Cypher* 0 *to the analogous Office in Univerſal Arithmetic, which it is always known to occupy in Common Arithmetic. As that Character is here diſcovered to be of the ſame Uſe for determining the Ratio of different Quantities to one another in their Ultimate State, which is made of it for aſſigning to each Figure its proper Place in abſolute Numbers. From whence will abundantly appear, what good Reaſons the ſagacious* [*] *Author of the* Analyſt *had to condemn the foregoing Methods for the ſame Ends; how malapertly ſoever that Work has been rather cavilled at, than judiciouſly anſwered by any.*

It is here demonſtrated, that every Equation, according to the Conditions it includes, has a certain Ultimate Limit, *which its unknown Quantities can by no Means either exceed, or fall ſhort of. A Property, which (however heretofore unknown, or, at leaſt, not ſufficiently conſidered) is as eſſential to thoſe Quantities, while under ſuch Conditions, as Extenſion or Diviſibility itſelf. And the Determinant of this Limit; which (till a more proper Name be found) I beg leave to call an* Ultimator, *is the very ſame with what has been ſtiled a* Differential *by ſome, and a* Fluxion *by others. Both which laſt Terms will be diſcovered to be much about as expreſſive of the true Meaning of the Thing, as the Sound of a Trumpet wou'd be to convey the Idea of Scarlet to a Man born blind.*

---

[*] Dr. *Berkley*, Biſhop of *Cloyne* in *Ireland*.

The Name Differential † *taken in this Senſe, is proved to imply a flat Contradiction; and that of* Fluxion *the ſame Way applied, can poſſibly have no Meaning at all. For pray what Kind of Meaning can be made of ſuch unintelligible Jargon as this:*

> “ *We are taught, that tho'* Fluxions *cannot be called Quantities themſelves, yet they have the* Ratio *of Quantities. And, becauſe it is impoſſible for any one to underſtand how Things can have the* Ratio *to one another of Quantities, without conſidering thoſe Things at the ſame Time as Quantities themſelves, and ſome Way the Subjects of Extenſion and Diviſibility; therefore to make us underſtand this* Ratio, *it is ſaid to be that of the* firſt naſcent Increments, *or* laſt evaneſcent Decrements of a Quantity. *That is, according to the Explanation at the ſame Time given of theſe Terms;* It is the Ratio of an Increaſe without any Increaſe, *or* of a Decreaſe without a Decreaſe; and all this perform'd in Time and no Time by a Motion and no Motion.”

But the ſuperlative Impropriety of the Word Fluxion, *when applied to this Purpoſe, will fully appear; when we come to conſider, that it is put to expreſs an Idea, which ariſes from the Contemplation of Quantities purely as Quantities: that is, in the ſame abſtract Manner, as they are the proper Subject of* Algebra, *excluſive of every other Conſideration; and conſequently have not the leaſt Regard to Time or Motion, which are neceſſarily implied in a* Fluxion. *And the eſſential Property of a* Fluxion *is certainly excluded, after the moſt ſingular Manner, in the Idea of a Quantity conſidered at its* Ne plus ultra: *that is, in other Terms, when it is in a State, where all Poſſibility of ſuch imaginary* Flux *is taken from it. So that the Term* Fluxion, *when uſed to this Purpoſe, if it have any Meaning at all, is as contrary to the true, as Darkneſs is to Light.*

Such are the Abſurdities, which muſt neceſſarily follow from abſurd Principles. And no better have been the Reaſonings of thoſe, who were otherwiſe the moſt intelligent Perſons, when they wou'd make us underſtand, what they cannot underſtand themſelves. For

---

† See *Page* 30, 31, 49.

ſuch I affirm theſe Notions of a Fluxion *and a* Differential *to be. Things, which can poſſibly have no Exiſtence in Nature: But ſeem purely to have been brought into the Imagination by a mixed Multiplication of* finite Quantities *and* Indiviſibles *together, according to their ſuppoſed Generation of Magnitudes; which accidentally threw the ingenious Inventor upon thoſe Conſequences, that naturally flow from the Reduction of an Equation to its* Ultimum: *While, it is plain, he in the mean Time had not the leaſt Suſpicion of the real Foundation, upon which his own Superſtructure muſt reſt.*

I am not unapprehenſive, what a Reception this Performance in all Likelihood will meet with from ſuch as have been long wedded either to the Notion of Fluxions, *or* Differentials. *It holds not only in Religion and Morality, but even in the Mathematicks themſelves; that when a Truth does not immediately appear, tho' it be never ſo demonſtrable in itſelf; yet, if Mens Prejudices be ſuch, as not to permit them to examine into the Grounds of it, they effectually preclude themſelves from all Means of being ever undeceived.*

How far the Force of Uſe and the Examples of great Authority will go, the ingenious Mr. Hodgſon *affords us a remarkable Inſtance in the Introduction to his Book of* Fluxions, *Page* 11. *where he ſays:* Whatever may have induced ſome People to imagine, that Sir *Iſaac Newton*, after having revolved in his Mind the Doctrine of *Fluxions* for forty Years, has not delivered himſelf ſo clearly as he might have done, and have therefore endeavoured to explain it better; for my Part, I muſt confeſs, I cannot ſee the leaſt Shadow of Reaſon for ſuch an Objection, and Attempt. Nor can I think there is any more Difficulty in conceiving or forming an adequate Notion of a naſcent or evaneſcent Quantity, than there is of a Mathematical Point; which, tho' it be void of Magnitnde, is the Foundation, or Root, from whence all Kinds of Magnitudes take their Riſe. *Now, to inſtance in the Generation of a Line by the Motion of a Point; this Gentleman gives us to underſtand, that in that Caſe, by a* naſcent *or* evaneſcent Quantity, *he means only the Tendency of that Point to move indefinitely, ſometimes according to one Direction, and ſometimes according to another. This I allow is eaſy enough to be conceived: But of what Uſe the unknown Manner of this Tendency is, to make us apprehend any Thing elſe by it, but ſtill a pure Point, I cannot conceive. And, as to any Signification it can have for determining the Length of the Line thus generated, he might as* *well have ſubſtituted in its Place, the Tendency of the ſame Point to produce ſuch a Colour, ſuch a Sound, ſuch a Taſte, or ſuch a Smell. And the ſame may be ſaid concerning the Generation of a Superficies by a Line, or of a Solid by a Superficies: See* Pages 33, 31, 35—50, *and* 71, 72 *of the following Work.*

I profeſs myſelf to have as due a Veneration as any Man for both the great Sir Iſaac Newton, *and his Cotemporary the celebrated Mr.* Leibnitz; *whichſoever of the two was the firſt Inventor of the Superſtructure already built upon this Foundation. But yet I hope I ſhall be excuſed in diſſenting from the higheſt human Authority, where I have Demonſtration on my Side, and conſequently (which is the Caſe here) where that Authority muſt want it.*

In the Whole, the diligent Student will perceive a new and large Field laid open for him to employ his Invention upon; which, I doubt not, will in Time be carried forward, (by ſuch as have more Leiſure and Ability than I have) to ſtill higher Improvements, much beyond whatever has been heretofore known. And to excite his Curioſity I preſume I may with Truth affirm, that (unleſs ſome Perſon in the mean Time has made the ſame Diſcovery, unknown to me;) the learned World in general are yet as ignorant of it, as the Americans *were at firſt how the* Europeans *cou'd convey their Thoughts to one another at a Diſtance by Pieces of Paper.*

Whatever Imperfection may be found through the Body of the Work, I hope will be excuſed, not only upon Account of human Fallibility in its beſt Condition, but alſo upon Account of my own uncommon Embarraſſment, during the whole Performance, under an Accumulation of well nigh all thoſe Difficulties together, which are the greateſt Diſcouragement to Study. Difficulties, which, I am pretty ſure, were they perfectly known, wou'd make it ſcarce thought credible by thoſe who are acquainted with this Kind of Study; that a Work conſiſting wholly of ſuch a Chain of Inventions, entirely new to himſelf, cou'd ever be brought to the Height it is here carried to, under ſo many Inconveniences. And yet, I hope, the Learner has enough to teach him how to anſwer every Thing anſwerable, *which occurs in any Writer upon Fluxions hitherto: Only with this Difference, that it is here done with much more Certainty and Exactneſs.*

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ERRATA.

The Reader is deſired to correct the following *Errata* with his Pen; where note, that the running Title is never taken as a Line; and all fractional Expreſſions are taken in the ſame Line.

PAGE 6. line the laſt but one, for $+r+r$ write $+r+s$. P. 8. l. 5, for $\frac{n+x-2}{x-2} \times \frac{n+x-3}{n-3}$ write $\frac{n+x-3}{x-2} \times \frac{n+x-4}{x-3}$. P. 9. l. laſt but one, *affirmative.* P. 10. l. 9, write $\frac{2n-1}{3n}$. ibid l. laſt after $\frac{4n+1}{5n}$ inſert $\frac{5n+1}{6n}$. P. 13. l. 3, for $\frac{m-1}{2}$ write $\frac{m-1}{4}$, and for $\frac{m-3}{4}$ write $\frac{m-3}{2}$. P. 16. l. 13. for $\frac{a^v}{c^c}$ write $\frac{a^v}{c^v}$, l. 14. for $v+d-1$ write $v+2d-1$. P. 17. l. 8, write $\frac{z^v}{c^{nv}}$, l. laſt, $nd \times \overline{v+3d-3}$. P. 20. l. 8. for $=a^6$ write $a^6=$; l. 17, for $+n$ write $+u$. P. 21. l. 4, for $4-2$ write $4-u$. P. 24. l. laſt, $14a^2$. P. 25. l. 2, $z=Z$. P. 28. l. 17. for 66 write 56. P. 32. l. 4 write $\frac{3}{10}z^5$. P. 33. l. 6, for x^{12} write x^{11}. P. 34. l. 6. for 982, write 983. P. 36. *affirmative.* P. 41. l. 7. for a write a^a. P. 43. l. laſt but three, for x^2 write z^2; ibid. l. laſt but one for *dividing* write *Then dividing*. P. 44. l. 14. for $\frac{x^n}{z^n}$ write $\frac{x^n}{z}$; ibid l. 18, write *Multiply*; ibid l. laſt but one, for *by* write *be*. P. 45. l. 1. for xxx, write x^2x. P. 46. l. 25. for x^n write c^n. P. 47. l. 10, 11, 12, for z^{y^x} writ z^{y^x}. P. 48.

P. 56.

E R R A T A.

l. 10. for *Laws*, write *neceſſary Laws*. P. 49. l. 29, for *right* write a. P. 51. l. l. 10, for $2rr^{o}$ write $2rv^{o}$. P. 52. l. 2. for s write S; l. 7. for 1652 write 1152. P. 53. l. 18, write $\frac{ss^{o}}{\overline{cc-ss}^{\frac{3}{2}}}$; *ibid.* l. laſt but one, for $ss^{o}v^{o}v^{o}$, write $s^{o}s^{o}+v^{o}v^{o}$. P. 54. l. 1, write $\frac{cc}{cc-\overline{r-v}^{2}}v^{o}v^{o}$. P. 55. l. 14, write $\overline{cc-ss}^{\frac{1}{2}}$. P. 56. l. 11, write $\frac{qaa}{2r}$. P. 57. l. 24, for $s^{o}e$ write $s^{o}=e$. P. 58. l. 7. write $6b^{3}v^{2}v^{o}$; l. 8. write $\frac{2b^{3}v^{o}}{\overline{bb+vv}^{3}}$; l. laſt but one, write ~~$\overline{2bp^{2}3-ba^{2}-a}^{3}$~~ $\overline{2bp^{2}-3ba^{2}-a^{3}}$; l. laſt write $\overline{b+p}\times p^{2}$. P. 59. l. 6. for *a is*, write a^{o} is; l. 21. for an^{n-1}, write na^{n-1}; l. 22, for $\overline{n-1}\times\overline{n-2}$ write $n\times\overline{n-1}\times\overline{n-2}$; l. 24, for $a^{2}a^{o}$ write $a^{oo}a^{oo}$; l. laſt but one for e^{oo}, write a^{oo}. P. 62. l. 15, for yy^{o} write $y^{o}y^{o}$. P. 63. l. 5, for v^{o} write u^{o}; *ibid.* l. 10, for $y^{2}y^{o}$, write $y^{2}\overset{o}{y^{o}}$; l. 11, dele $-cy\overset{o}{y^{o}}$; l. 13, for c^{o}, write c^{2}. P. 64. l. laſt but three, *after that fractional Expreſſion* for, write $=0$, or z. P. 65. l. 1, for $MC\overline{v^{o}v^{o}+s^{o}s^{o}}^{\frac{3}{2}}=q$, write $MC=\frac{\overline{v^{o}v^{o}+s^{o}s^{o}}^{\frac{3}{2}}}{-v^{o}s^{oo}}=q$. P. 68. l. 9, for $\overline{2cv-vv}^{\frac{1}{2}}$, write $2cv-vv$. P. 73. l. 1, write $2c^{3}$; l. 2, write $327\overset{6}{\wedge}8$; l. 3. write $\frac{11}{10c^{3}}s^{5}$; l. 4, write 35816, and 622592. P. 75. l. 10. l. 11. for *theſe* write *their*; for *their* write *theſe*; l. 16, for $\frac{b}{q}$ write

$\frac{q}{b}$

E R R A T A.

$\frac{q}{b}$. P. 76. dele 3^r. P. 80. l. 4, write *Mixed*. P. 82. l. laſt for $\overline{x+\dot{s}}$ write $\overline{xl+\dot{s}}$. P. 85. l. 2, for b q, write b p; l. laſt but five write *fixed*, and s s for s. P. 88. l. 13, for $5x^6$ write $5x^4$. P. 92. l. 2, for u^3 write v^3; l. 7, for $v^{\frac{1}{1}}$, write $v^{\frac{3}{2}}$; l. 8, write $\frac{1}{3}+$; l. laſt but one, for 12772520 write 12779520. P. 102. l. 1. write $\overline{\tfrac{1}{2}b s^{\frac{1}{2}} - s^{\frac{3}{2}}} \times \frac{s^{\bullet}}{\overline{b - s^{\frac{1}{2}}}}$. P. 105. l. laſt but five write $\frac{z^3}{5n^3}$. P. 108. l. 10. before *of* write, *ſo is the Ultimator* y° *to the Ultimator* $\frac{y^{\circ}}{y^2} \times \overline{n^4+y^4}^{\frac{1}{2}}$; laſt but three, for 57 write 5 y. P. 112. write $y = KA =$. P. 120. l. laſt but five, for $AC - AC$, write $VC - AC$. P. 129. l. laſt but two, for $-$ write $=$.

THE

A NEW ACQUISITION TO Mathematical Literature.

SECT. I.

Some neceſſary Premonitions from common Algebra.

I. WHEN any Number of Coefficients is taken as one Coefficient; if the firſt be a Negative, then by changing all the Signs of the following Ones the Whole may be repreſented as a negative Quantity. Thus, $-10a+17a-3a$ is the ſame with $-\overline{10-17+3}\times a=4a$. So $\frac{n-m}{b}$ is the ſame with $-\frac{m-n}{b}$. For to take away a Negative is the ſame with adding an Affirmative.

II. Multiplication or Diviſion, by the ſame Letters, is performed by adding or ſubtracting their Exponents reſpectively. Thus $a^5\times a^2=a^{5+2}=a^7$; $\frac{a^5}{a^2}=a^{5-2}=a^3$, and $\frac{a^2}{a^5}=a^{2-5}=a^{-3}$; alſo $\frac{a}{a^3}=a^{1-3}=a^{-2}$, and $\frac{a^n}{a^n}=a^{n-n}=a^0=1$.

III. To divide any Quantity a, by another Quantity c, is all one as to multiply the ſame Quantity a by $\frac{1}{c}$; and to multiply any Quantity a by another Quantity c is all one as to divide the ſame Quantity a by $\frac{1}{c}$. That is $\frac{a}{c}=a\times\frac{1}{c}$, and $\frac{1}{c}\big)\,a=ac$.

IV. Any two Quantities with different fractional Exponents may be made to have the ſame Exponent, by bringing their Exponents to have the ſame Denominator, and then raiſing each Quantity to the Power, which the Numerator of its new Exponent ſhall direct. And ſuch Quantities, as are thus brought to have the ſame Exponent, may have their Products and Quotients expreſſed with that Exponent common to both. Thus, $a^{\frac{1}{2}}\times c^{\frac{2}{3}}=a^{\frac{3}{6}}\times a^{\frac{4}{6}}=\overline{a^3c^4}^{\frac{1}{6}}$, and $a^{\frac{1}{2}}\big)\,\overline{a^3c^4}^{\frac{1}{6}}=a^{\frac{3}{6}}\big)\,\overline{a^3c^4}^{\frac{1}{6}}=\overline{aaa}^{\frac{1}{6}}\big)\,\overline{aaacccc}^{\frac{1}{6}}=c^{\frac{4}{6}}=c^{\frac{2}{3}}$. So $a\times c^{\frac{1}{2}}=\overline{aac}^{\frac{1}{2}}$, $a\big)\,\overline{aac}^{\frac{1}{2}}=c^{\frac{1}{2}}$. So $a^{\frac{1}{3}}\times c^{\frac{2}{5}}=\overline{a^5c^6}^{\frac{1}{15}}$, and $a^{\frac{1}{3}}\big)\,\overline{a^5c^6}^{\frac{1}{15}}=c^{\frac{2}{5}}$, &c.

V. When therefore the Root of a whole Multiple cannot be had, the Root of ſome Submultiple of it perhaps may. Thus $\overline{aabc}^{\frac{1}{2}}=a\times\overline{bc}^{\frac{1}{2}}$. So $\overline{48}^{\frac{1}{2}}=4^2\times3^{\frac{1}{2}}=\overline{4\times3}^{\frac{1}{2}}$. So $6\times\frac{75}{98}\Big|^{\frac{1}{2}}=\frac{36\times75}{98}\Big|^{\frac{1}{2}}=\frac{36\times25\times3}{49\times2}\Big|^{\frac{1}{2}}=\overline{\frac{36\times25}{49\times4}\times6}^{\frac{1}{2}}=\frac{6\times5}{7\times2}\times6^{\frac{1}{2}}=\frac{3\times5}{7}\times6^{\frac{1}{2}}$. So $\overline{aaaz}^{\frac{1}{4}}=\frac{aaaaz}{a}\Big|^{\frac{1}{4}}=a\times\frac{z}{a}\Big|^{\frac{1}{4}}$. So $\overline{16a^4+8a^3b}^{\frac{1}{3}}=\overline{8a^3\times\overline{2a+b}}^{\frac{1}{3}}=2a\times\overline{2a+b}^{\frac{1}{3}}$, &c. And whenever two or more Irrationals can by theſe Means be brought to the ſame Quantity, under the ſame Exponent they may be added or ſubtracted. Thus $\overline{48}^{\frac{1}{2}}+\overline{75}^{\frac{1}{2}}=\overline{16\times3}^{\frac{1}{2}}+\overline{25\times3}^{\frac{1}{2}}=4\times3^{\frac{1}{2}}+5\times3^{\frac{1}{2}}=9\times3^{\frac{1}{2}}$. So $\overline{175}^{\frac{1}{2}}-\overline{28}^{\frac{1}{2}}=\overline{25\times7}^{\frac{1}{2}}-\overline{4\times7}^{\frac{1}{2}}$

$\overline{4\times7}^{\frac{1}{2}} = \overline{5-2}\times7^{\frac{1}{2}} = 3\times7^{\frac{1}{2}}$. So $\overline{135}^{\frac{1}{3}} - \overline{40}^{\frac{1}{3}} = \overline{27\times5}^{\frac{1}{3}} - \overline{8\times5}^{\frac{1}{2}} = 5^{\frac{1}{3}}$. &c.

VI. When a Diviſor cannot be had exactly in a Dividend, the Quotient will be expreſſed by an infinite Series, as in the two following Examples:

$$
\begin{array}{l|l}
a+e)\,c\,\left(\frac{a}{c}-\frac{ce}{a^2}+\frac{ce^2}{a^3}-\frac{ce^3}{a^4}+\&c. & a-e)\,c\,\left(\frac{c}{a}+\frac{ce}{a^2}+\frac{ce^2}{a^3}+\frac{ce^3}{a^4}+\&c.\right. \\
\quad c+\frac{ce}{a} & \quad c-\frac{ce}{a} \\
\hline
\quad -\frac{ce}{a} & \quad +\frac{ce}{a} \\
\quad -\frac{ce}{a}-\frac{ce^2}{a^2} & \quad +\frac{ce}{a}-\frac{ce^2}{a^2} \\
\hline
\qquad +\frac{ce^2}{a^2} & \qquad +\frac{ce^2}{a^2} \\
\qquad +\frac{ce^2}{a^2}+\frac{ce^3}{a^3} & \qquad +\frac{ce^2}{a^3}-\frac{ce^3}{a^3} \\
\hline
\qquad\qquad -\frac{ce^3}{a^3}\ \&c. & \qquad\qquad +\frac{ce^3}{a^3}\ \&c.
\end{array}
$$

Therefore the Quotient ariſing from the Diviſion of any Quantity *c* by the Diviſor $a \pm e$ is equal to $\frac{c}{a}$ multiplying the whole Series $1 \mp \frac{e}{a} + \frac{e^2}{a^2} \mp \frac{e^3}{a^3} + \frac{e^4}{a^4} \mp \frac{e^5}{a^5} + \frac{e^6}{a^6} \mp$ &c.

VII. And ſuch Ranks are multiplied and divided as follows:

$$\begin{array}{llllllllll}
\textit{Multiply} & a & -\frac{e^2}{2a} & -\frac{e^4}{8a^3} & -\frac{e^6}{16a^5} & -\frac{5e^8}{128a^7} & -\frac{7e^{10}}{256a^9} & -\frac{21e^{12}}{1024a^{11}} & - & \&c. \\
\textit{By} & a & -\frac{e^2}{2a} & -\frac{e^4}{8a^3} & -\frac{e^6}{16a^5} & -\frac{5e^8}{128a^7} & -\frac{7e^{10}}{256a^9} & -\frac{21e^{12}}{1024a^{11}} & - & \&c. \\
\hline
 & a^2 & -\frac{e^2}{2} & -\frac{e^4}{8a^2} & -\frac{e^6}{16a^4} & -\frac{5e^8}{128a^6} & -\frac{7e^{10}}{256a^8} & -\frac{21e^{12}}{1024a^{10}} & - & \&c. \\
 & & -\frac{e^2}{2} & +\frac{e^4}{4a^2} & +\frac{e^6}{16a^4} & +\frac{e^8}{32a^6} & +\frac{5e^{10}}{256a^8} & +\frac{7e^{12}}{512a^{10}} & + & \&c. \\
 & & & -\frac{e^4}{8a^2} & +\frac{e^6}{16a^4} & +\frac{e^8}{64a^6} & +\frac{e^{10}}{128a^8} & +\frac{5e^{12}}{1024a^{10}} & + & \&c. \\
 & & & & -\frac{e^6}{16a^4} & +\frac{e^8}{32a^6} & +\frac{e^{10}}{128a^8} & +\frac{e^{12}}{256a^{10}} & + & \&c. \\
 & & & & & -\frac{5e^8}{128a^6} & +\frac{5e^{10}}{256a^8} & +\frac{5e^{12}}{1024a^{10}} & + & \&c. \\
 & & & & & & -\frac{7e^{10}}{128a^8} & +\frac{7e^{12}}{512a^{10}} & + & \&c. \\
 & & & & & & & -\frac{21e^{12}}{1024a^{10}} & + & \&c. \\
\hline
\textit{Product} & a^2 & -e^2 & * & * & * & * & * & &
\end{array}$$

Divisor *Dividend* *Quotient*

$$a+\frac{1}{3}a^3+\frac{2}{15}a^5+\frac{17}{315}a^7+\&c.\Big)\,a-\frac{1}{6}a^3+\frac{1}{120}a^5-\frac{1}{5040}a^7+\&c.\Big(\,1-\frac{1}{2}a^2+\frac{1}{24}a^4-\frac{1}{720}a^6+\&c.$$

$$\begin{array}{r}
a+\frac{1}{3}a^3+\frac{2}{15}a^5+\frac{7}{315}a^7+\&c. \\
\hline
-\frac{1}{2}a^3-\frac{1}{8}a^5-\frac{13}{240}a^7-\&c. \\
-\frac{1}{2}a^3-\frac{1}{6}a^5-\frac{2}{30}a^7-\&c. \\
\hline
+\frac{1}{24}a^5+\frac{1}{80}a^7+\&c. \\
+\frac{1}{24}a^5+\frac{1}{72}a^7+\&c. \\
\hline
-\frac{1}{720}a^7\ \&c.
\end{array}$$

SECT. II.

Of REMOVES.

EVERY Rank or Series of Terms, proceeding by the continual Addition or Subtraction of the ſame common Difference, may be called a Series of the firſt Remove; for the Sum of every Number of which putting *A*, it will form a Series of the ſecond Remove; for the Sum of every Number of which putting *B*, it will form a Series of the third Remove; for the Sum of every Number of which putting *C*, it will form a Series of the fourth Remove; for the Sum of every Number of which putting *D*, it will form a Series of the fifth Remove; and ſo on to *E* for the ſixth Remove; *F* for the ſeventh, *&c.* For Example, ſuppoſe $5=n$ the Number of Terms, $l=$ the leaſt Term, $g=$ the greateſt, and $d=$ the common Difference: Then, if we proceed by a continual Addition from l the firſt Term.

The direct Scale or Aſcent is.

$$\begin{array}{llllll}
l, & l+d, & l+2d, & l+3d, & l+4d & =g \\
l, & 2l+d, & 3l+3d, & 4l+6d, & 5l+10d & =A \\
l, & 3l+d, & 6l+4d, & 10l+10d, & 15l+20d & =B \\
l, & 4l+d, & 10l+5d, & 20l+15d, & 35l+35d & =C \\
\&c. & \&c. & \&c. & \&c. & \&c. & \&c.
\end{array}$$

In which Scale it will be found, that whenever n be (if we put $p=$ the Coefficient of d in A, $q=$ the Coefficient of d in B, $r=$ the Coefficient of d in C, $s=$ that of d in D, $t=$ that of d in E, *&c*,) we have in every Number of Terms univerſally $l+nd-d=g$, $nl+pd=A$, $\overline{n+p}\times l+qd=B$, $\overline{n+p+q}\times l+rd=C$, $\overline{n+p+q+r}\times l+sd=D$, $\overline{n+p+q+r+s}\times l+td=E$, &c. Whence $l=g-nd+d$.

If therefore we proceed again by a continual Addition from $g-nd+d=l$.

The retrogade Scale is,

$g-nd+d,\ g-nd+2d,\ g-nd+3d,\ g-nd+4d,\ g-nd+5d=g$
$g-nd+d,\ 2g-2nd+3d,\ 3g-3nd+6d,\ 4g-4nd+10d,\ 5g-5nd+15d=A$
$g-nd+d,\ 3g-3nd+4d,\ 6g-6nd+10d,\ 10g-10nd+20d,\ 15g-15nd+35d=B$
$g-nd+d,\ 4g-4nd+5d,\ 10g-10nd+15d,\ 20g-20nd+35d,\ 35g-35nd+70d=C$

In which Scale it will alſo appear, whatever n be, that the Coefficient of g in each Series, is reſpectively the ſame with that of l in the former Scale. Alſo the Coefficient of d in the firſt Series, is $-n+n=0$. The Coefficient of d in A is $-nn+n+p=-p$; whence $p=\frac{n-1}{2}\times n$. The Coefficient of d in B is $-n\times\overline{n+p}+n+p+q=-2q$; whence $q=\frac{n-1}{3}\times n\times\frac{n+1}{2}$. The Coefficient of d in C is $-n\times\overline{n+p+q}+n+p+q+r=-3r$; whence $r=\frac{n-1}{4}\times n\times\frac{n+1}{2}\times\frac{n+2}{3}$. And by thus proceeding we have $s=\frac{n-1}{5}\times n\times\frac{n+1}{2}\times\frac{n+2}{3}\times\frac{n+3}{4}$, $t=\frac{n-1}{6}\times n\times\frac{n+1}{2}\times\frac{n+2}{3}\times\frac{n+3}{4}\times\frac{n+4}{5}$, &c. And univerſally putting $N=$ the Coefficient, and $x=$ the Number of the Remove, we have $N=\frac{n-x}{x}\times\frac{n+x-2}{x-1}\times\frac{n+x-3}{x-2}\times\frac{n+x-4}{x-3}\times\frac{n+x-5}{x-4}$, &c. till the Diviſor becomes $x-x=0$; and conſequently that Factor with all the following ones are rejected. Hence $A=nl+pd=2\overline{l+nd-d}\times\frac{n}{2}$, $B=\overline{n+p}\times l+qd=\overline{3l+nd-d}\times\frac{n}{2}\times\frac{n+1}{3}$, $C=\overline{n+p+q}\times l+rd=$ $\overline{4l+}$

$\overline{4l+nd-d}\times\frac{n}{2}\times\frac{n+1}{3}\times\frac{n+2}{4}$, *&c.* Or univerſally putting Q for *A*, *B*, *C*, *D*, *&c.* and x equal to the Number of the Remove; we have $Q=\frac{lx+nd-d}{x}\times\frac{n+x-2}{x-1}\times\frac{n+x-3}{x-2}\times\frac{n+x-4}{x-3}\times\frac{n+x-5}{x-4}$, *&c.* till the Diviſor becomes $x-x=0$. THEOREM I. Hence if $l=d=1$, we have $\frac{n+x-1}{x}\times\frac{n+x-2}{x-1}\times\frac{n+x-3}{x-2}\times\frac{n+x-4}{x-3}\times\frac{n+x-5}{x-4}$, *&c.* $=Q$.

SECT. III.

Of the Compoſition *and* Reſolution *of* Powers.

IF it be required to raiſe the Binomial $c\pm a$ to its univerſal n Power, by a continual Multiplycation of $\overline{c\pm a}$ by it ſelf, we have,

$\overline{c\pm a}^{1}=c\pm a$

$\overline{c\pm a}^{2}=c^2\pm 2ca+a^2$

$\overline{c\pm a}^{3}=c^3\pm 3c^2a+3ca^2\pm a^3$

$\overline{c\pm a}^{4}=c^4\pm 4c^3a+6c^2a^2\pm 4ca^3+a^4$

$\overline{c\pm a}^{5}=c^5\pm 5c^4a+10c^3a^2\pm 10c^2a^3+5ca^4\pm a^5$, &c.

Whence it appears, that the *Unciæ* (or numeral Coefficients) of all the ſecond Terms are a Series of natural Numbers, 1, 2, 3, 4, 5, *&c.* whoſe Number of Terms and laſt Terms are each equal to n, the Exponent of the Power. The *Uncia* of each third Term is a Number *A* of the ſecond Remove raiſed from the ſame Series, till the Number of Terms be $n-1$ (whence *A* $= n\times\frac{n-1}{2}$): Of the fourth Term, a Number *B* of the third Remove raiſed from the ſame, till the Number of Terms be $n-2$ (whence $B = n\times\frac{n-1}{2}\times\frac{n-2}{3}$): Of the fifth Term, a Number *C* of the fourth Remove raiſed from the ſame, till the Number of Terms be $n-3$ (whence $C = n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}$): Of the ſixth Term, a Number *D* of the fifth Remove raiſed from the ſame, till the Number of Terms be $n-4$ (whence $D = n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}\times\frac{n-4}{5}$): &c.

Conſequently (if $p=\frac{n-1}{2}$, $q=\frac{n-2}{3}$, $r=\frac{n-3}{4}$, $s=\frac{n-4}{5}$, $t=\frac{n-5}{6}$, &c.) we have $\overline{c\pm a}^{\,n} = c^{n} \pm nc^{n-1}a + \frac{n-1}{2}\times nc^{n-2}a^{2} \pm \frac{n-2}{3}npc^{n-3}a^{3} + \frac{n-3}{4}npqc^{n-4}a^{4} \pm \frac{n-4}{5}\times npqrc^{n-5}a^{5} +$ &c. Or (becauſe $c^{-1}=\frac{1}{c}$, $c^{-2}=\frac{1}{c^{2}}$, $c^{-3}=\frac{1}{c^{3}}$, &c.) $\overline{c\pm a}^{\,n} = c^{n}$ multiplying the Series $1 \pm n\frac{a}{c} + \frac{n-1}{2}n\frac{a^{2}}{c^{2}} \pm \frac{n-2}{3}np\frac{a^{3}}{c^{3}} + \frac{n-3}{4}npq\frac{a^{4}}{c^{4}} \pm \frac{n-4}{5}npqr\frac{a^{5}}{c^{5}} + \frac{n-5}{6}npqrs\frac{a^{6}}{c^{6}} \pm$ &c. Otherwiſe $\overline{c\pm a}^{\,n} = c^{n}$ multiplying the Series $1 \pm n\times\frac{a}{c} + n\times\frac{n-1}{2}\times\frac{a^{2}}{c^{2}} \pm n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{a^{3}}{c^{3}} + n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}\times\frac{a^{4}}{c^{4}} \pm n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}\times\frac{n-4}{5}\times\frac{a^{5}}{c^{5}} + n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}\times\frac{n-4}{5}\times\frac{n-5}{6}\times\frac{a^{6}}{c^{6}} \pm$ &c.

THEOREM II.

Which Theorem, when n is a finite Number (whether integer or fracted, affirmative or negative) may ſerve for raiſing all Powers, and extracting all Roots univerſally.

If we ſubſtitute $-n$ for n (whence c^n becomes c^{-n} or $\frac{1}{c^n}$) we have $\overline{c\pm a}^{\,-n}$ or $\frac{1}{\overline{c\pm a}^{\,n}} = \frac{1}{c^n}$ multiplying the Series $1 \mp n \times \frac{a}{c} + n \times \frac{n+1}{2} \times \frac{a^2}{c^2} \mp n \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{a^3}{c^3} + n \times \frac{n-1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times \frac{a^4}{c^4} \mp n \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times \frac{n+4}{5} \times \frac{a^5}{c^5} + n \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times \frac{n+4}{5} \times \frac{n+5}{6} \times \frac{a^6}{c^6}$, &c. THEOREM III.

If we ſubſtitute in the ſame ſecond Theorem $\frac{1}{n}$ for n, z for a, and c^n for c there (whence c^{2n} muſt be put for c^2 there, c^{3n} for c^3, c^{4n} for c^4, &c.) We have $\overline{c^n \pm z}^{\,\frac{1}{n}} = c$ multiplying the the Series $1 \pm \frac{1}{n} \times \frac{z}{c^n} - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{z^2}{c^{2n}} \pm \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{z^3}{c^{3n}} - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{z^4}{c^{4n}} \pm \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n} \times \frac{z^5}{c^{5n}} - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n} \times \frac{5n-1}{6n} \times \frac{z^6}{c^{6n}} \pm$, &c. THEOREM IV.

So by the ſame Procedure, if we ſubſtitute $-\frac{1}{n}$ for n there, we have $\overline{c^n \pm z}^{\,-\frac{1}{n}}$ or $\frac{1}{\overline{c^n \pm z}^{\,\frac{1}{n}}} = \frac{1}{c}$ multiplying the Series $1 \mp \frac{1}{n} \times \frac{z}{c^n} + \frac{1}{n} \times \frac{n+1}{2n} \times \frac{z^2}{c^{n2}} \mp \times \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{z^3}{c^{3n}} + \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{2n} \times \frac{3n+1}{4n} \times \frac{z^4}{c^{4n}} \mp \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{3n+1}{4n} \times \frac{4n+1}{5n} \times \frac{z^5}{c^{5n}} + \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{3n+1}{4n} \times \frac{4n+1}{5n} \wedge \times \frac{z^6}{c^{6n}} \mp$ &c. THEOREM V.

$\wedge \times \frac{5n+1}{6n}$

SECT. IV.

Of the Reciprocation of Powers with their Roots.

IT appears from the ſecond and fourth Theorems above; that if $\overline{c\pm a}^{\,n} = c^n \pm z$, and conſequently $c \pm a = \overline{c^n \pm z}^{\,\frac{1}{n}}$; we have $c^n \pm z$ equal to the Series of the former Theorem, and $c \pm a$ equal to the Series of the latter Theorem. Therefore by taking away c^n from both Sides of the former Equation, and c from both Sides of the latter; we have z equal to c^n multiplying the Series $n \times \frac{a}{c} \pm n \times \frac{n-1}{2} \times \frac{a^2}{c^2} + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{a^3}{c^3} \pm n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{a^4}{c^4} + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{a^5}{c^5} \pm n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6}\,\frac{a^6}{c^6} \pm$ &c. and a equal to c multiplying the Series $\frac{1}{n} \times \frac{z}{c^n} \mp \frac{1}{n} \times \frac{n-1}{2n} \times \frac{z^2}{c^{2n}} + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{z^3}{c^{3n}} \mp \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{z^4}{c^{4n}} + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n} \times \frac{z^5}{c^{5n}} \mp \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n} \times \frac{5n-1}{6n} \times \frac{z^6}{c^{6n}} +$ &c. *Confectary*, 1. Which gives a Solntion to all Equations of the following Scale.

$c^0 a^1 = z$

$2c^1 a \pm a^2 = z$

$3c^2 a \pm 3c\,a^2 + a^3 = z$

$4c^3 a \pm 6c^2 a^2 + 4c\,a^3 \pm a^4 = z$

$5c^4 a \pm 10c^3 a^2 + 10c^2 a^3 \pm 5ca^4 + a^5 = z$, &c.

This

This Scale you ſee is the very ſame with that of the Compoſition of Powers in the foregoing Section. And the two Equations of this Concluſion between the Power and its Root, I conceive may be properly enough named their *Reciprocation*. In which the former Series may be diſtinguiſhed by calling it the *Compoſition* of a the Quantity ſought, or *Reſolutum* of z the Quantity given; and the latter Series, by calling it the *Reſolutum* of a the Quantity ſought, or *Compoſitum* of z the Quantity given.

Now if we omit every other (or d) Term in ever other (or d) Equation of the laſt Scale, and raiſe the *Unciæ* from the Series 1, 3, 5, 7, 9 11, &c. (or *v*, *v*+d, *v*+2d, *v*+3d, *v*+4d, *v*+5d, &c.) obſerving that every Power of a muſt have its *Unciæ*, a Number of the ſame Remove with its Exponent; we ſhall have the Scale for a ſecond Confectary.

$$c^{\circ}a^{1} = z$$

$$3c^{2}a \pm a^{3} = z$$

$$5c^{4}a \pm 5c^{2}a^{3} + a^{5} = z$$

$$7c^{6}a \pm 14c^{4}a^{3} + 7c^{2}a^{5} \pm a^{7} = z$$

$$9c^{8}a \pm 30c^{6}a^{3} + 27c^{4}a^{5} \pm 9c^{2}a^{7} + a^{9} = z$$

$$11c^{10}a \pm 55c^{8}a^{3} + 77c^{6}a^{5} \pm 44c^{4}a^{7} + 11c^{2}a^{9} \pm a^{11} = z, \text{ \&c.}$$

Put m= the Number, incluſively of the Equation from the firſt Equation $c^{\circ}a^{1} = z$, and n = the higheſt Exponent of a in the Equation whoſe Number is m: Whence (Sect. II.) n = 1+2m—2 (or v+md—d) =2m—1. Then in the Series of the ſecond Section equal to Q, by ſubſtituting ſucceſſively, 1, 3, 5, 7, 9, 11, &c. (or *v*, *v*+d, *v*+2d, *v*+3d, *v*+4d, *v*+5d, &c.) for *x*; and reſpectively m, m—1, m—2, m—3, m—4, m—5, &c. for the n there, making 1=l there, we have z= to the *Compoſitum* made up of c^{n} multiplying the Series $\overline{2m-1} \times \frac{a}{c}$

$\frac{a}{a}\pm\frac{2m-1}{3}\times\frac{m}{2}\times\overline{m-1}\times\frac{a^3}{c^3}+\frac{2m-1}{5}\times\frac{m+1}{4}\times\frac{m}{3}\times\frac{m-1}{2}\times\overline{m-2}\times\frac{a^5}{c^5}$ $\pm\frac{2m-1}{7}\times\frac{m+2}{6}\times\frac{m+1}{5}\times\frac{m}{4}\times\frac{m-1}{3}\times\frac{m-2}{2}\times\overline{m-3}\times\frac{a^7}{c^7}+\frac{2m-1}{9}$ $\times\frac{m+3}{8}\times\frac{m+2}{7}\times\frac{m+1}{6}\times\frac{m}{5}\times\frac{m-1}{4}\times\frac{m-2}{3}\times\frac{m-3}{2}\times\overline{m-4}\times\frac{a^9}{c^9}\pm$ $\frac{2m-1}{11}\times\frac{m+4}{10}\times\frac{m+3}{9}\times\frac{m+2}{8}\times\frac{m+1}{7}\times\frac{m}{6}\times\frac{m-1}{5}\times\frac{m-2}{4}\times\frac{m-3}{3}\times$ $\frac{m-4}{2}\times\overline{m-5}\times\frac{a^{11}}{c^{11}}+$&c. But $n=2m-1$ $(=v+md-d)$; whence $m=\frac{n+1}{2}\left(=\frac{n+d-v}{d}\right)$. Conſequently $z=c^n$ multiplying the Series $n\times\frac{a}{c}\pm n\times\frac{n-1}{4}\times\frac{n+1}{6}\times\frac{a^3}{c^3}+n\times\frac{n-1}{4}\times\frac{n+1}{6}\times\frac{n-3}{8}\times\frac{n+3}{10}\times\frac{a^5}{c^5}\pm n$ $\times\frac{n-1}{4}\times\frac{n+1}{6}\times\frac{n-3}{8}\times\frac{n+3}{10}\times\frac{n-5}{12}\times\frac{n+5}{14}\times\frac{a^7}{c^7}+n\times\frac{n-1}{4}\times\frac{n+1}{6}$ $\times\frac{n-3}{8}\times\frac{n+3}{10}\times\frac{n-5}{12}\times\frac{n+5}{14}\times\frac{n-7}{16}\times\frac{n+7}{18}\times\frac{a^9}{c^9}\pm n\times\frac{n-1}{4}\times\frac{n+1}{6}\times$ $\frac{n-3}{8}\times\frac{n+3}{10}\times\frac{n-5}{12}\times\frac{n+5}{14}\times\frac{n-7}{16}\times\frac{n+7}{18}\times\frac{n-9}{20}\times\frac{n+9}{22}\times\frac{a^{11}}{c^{11}}+$&c.

And (by ſubſtituting $\frac{1}{n}$ for n, z for a, and c^n for c, as before) we have the *Reſolutum* $a=\frac{c}{n}$ multiplying the Series $\frac{z}{c^n}\mp\frac{n-1}{4n}\times\frac{n+1}{6n}$ $\times\frac{z^3}{c^{3n}}+\frac{n-1}{4n}\times\frac{n+1}{6n}\times\frac{3n-1}{8n}\times\frac{3n+1}{10n}\times\frac{z^5}{c^{5n}}\times\mp\frac{n-1}{4n}\times\frac{n+1}{6n}\times\frac{3n-1}{8n}$ $\times\frac{3n+1}{10n}\times\frac{5n-1}{12n}\times\frac{5n+1}{14n}\times\frac{z^7}{c^{7n}}+\frac{n-1}{4n}\times\frac{n+1}{6n}\times\frac{3n-1}{8n}\times\frac{3n+1}{10n}\times\frac{5n-1}{12n}$ $\times\frac{5n+1}{14n}\times\frac{7n-1}{16n}\times\frac{7n+1}{18}\times\frac{z^9}{c^{9n}}\mp\frac{n-1}{4n}\times\frac{n+1}{6n}\times\frac{3n-1}{8n}\times\frac{3n+1}{10n}\times\frac{5n-1}{12n}\times$ $\frac{5n+1}{14n}\times\frac{7n-1}{16n}\times\frac{7n+1}{18n}\times\frac{9n-1}{20n}\times\frac{9n+1}{22n}\times\frac{z^{11}}{c^{11n}}+$ &c. *Conſectary,* 2. Which gives a Solution to all Equations of the laſt Scale. *

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* *The Learner may, if he pleaſes, omit what follows in this Section; as being of no Neceſſity for underſtanding the main deſign in* SECT. VII. *&c.*

And univerſally, if $v = 1$, and $n = 1 + md - d$, whence $m = \frac{n+d-1}{d}$, we have z equal to the *Compoſitum* made up of c^n multiplying the Series $\overline{1+md-d} \times \frac{a}{c} \pm \frac{1+md-d}{1+d} \times \frac{m+d-2}{d} \times \frac{m+d-3}{d-1} \times \frac{m+d-4}{d-2} \times \frac{m+d-5}{d-3}$ &c. $\times \frac{a^{1+d}}{c^{1+d}} + \frac{1+md-d}{1+2d} \times \frac{m+2d-3}{2d} \times \frac{m+2d-4}{2d-1} \times \frac{m+2d-5}{2d-2} \times \frac{m+2d-6}{2d-3}$ &c. $\times \frac{a^{1+2d}}{c^{1+2d}} \pm \frac{1+md-d}{1+3d} \times \frac{m+3d-4}{3d} \times \frac{m+3d-5}{3d-1} \times \frac{m+3d-6}{3d-2} \times \frac{m+3d-7}{3d-3}$ &c. $\times \frac{a^{1+3d}}{c^{1+3d}} + \frac{1+md-d}{1+4d} \times \frac{m+4d-5}{4d} \times \frac{m+4d-6}{4d-1} \times \frac{m+4d-7}{4d-2} \times \frac{m+4d-8}{4d-3}$ &c. $\times \frac{a_{1+4d}}{c^{1+4d}} \pm$ &c. Or reſtoring $\frac{n+d-1}{d}$ for m, we have z equal to the *Compoſitum* made up of c^n multiplying the Series $n \times \frac{a}{c} \pm \frac{n}{1+d} \times \frac{n+dd-1-d}{dd} \times \frac{n+dd-1-2d}{dd-d} \times \frac{n+dd-1-3d}{dd-2d} \times \frac{n+dd-1-4d}{dd-3d}$ &c. $\times \frac{a^{1+d}}{c^{1+d}} + \frac{n}{1+2d} \times \frac{n+2dd-1-2d}{2dd} \times \frac{n+2dd-1-3d}{2dd-d} \times \frac{n+2dd-1-4d}{2dd-2d} \times \frac{n+2dd-1-5d}{2dd-3d}$ &c. $\times \frac{a^{1+2d}}{c^{1+2d}} \pm \frac{n}{1+3d} \times \frac{n+3dd-1-3d}{3dd} \times \frac{n+3dd-1-4d}{3dd-d} \times \frac{n+3dd-1-5d}{3dd-2d} \times \frac{n+3dd-1-6d}{3dd-3d} \times$ &c. $\times \frac{a^{1+3d}}{a^{1+3d}} + \frac{n}{1+4d} \propto \frac{n+4dd-1-4d}{4dd} \times \frac{n+4dd-1-5d}{4dd-d}\ x\ \frac{n+4dd-1-6d}{4dd-2d} \times \frac{n+4dd-1-7d}{4dd-3d}$ &c. $\times \frac{a^{1+4d}}{c^{1+4d}} \pm$ &c. Conſequently (putting $\frac{1}{n}$ for n, z for a, and c^n for c, as before) we have the *Reſolutum* $a = \frac{c}{n}$ multiplying the Series $\frac{z}{c^n} \mp \frac{1}{1+d} \times \frac{1+n\times\overline{dd-1}-nd}{ndd} \times \frac{1+n\times\overline{dd-1}-2nd}{n\times\overline{dd-d}} \times \frac{1+n\times\overline{dd-1}-3nd}{n\times\overline{dd-2d}} \times \frac{1+n\times\overline{dd-1}-4nd}{n\times\overline{dd-3d}}$ &c. $\times \frac{z^{1+d}}{c^{n\times\overline{1+d}}} + \frac{1}{1+2d} \times \frac{1+n\times\overline{2dd-1}-2nd}{2ndd} \times \frac{1+n\times\overline{2dd-1}-3nd}{n\times\overline{2dd-d}} \times$

1+n

$$\frac{1+n\times\overline{2dd-1}-4nd}{n\times\overline{2dd-2d}}\times\frac{1+n\times\overline{2dd-1}-5nd}{n\times\overline{2dd-3d}}\ \&c.\times\frac{z^{1+2d}}{c^{n\times\overline{1+2d}}}\mp\frac{1}{1+3d}\times$$

$$\frac{1+n\times\overline{3dd-1}-3nd}{3ndd}\times\frac{1+n\times\overline{3dd-1}-4nd}{n\times\overline{3dd-d}}\times\frac{1+n\times\overline{3dd-1}-5nd}{n\times\overline{3dd-2d}}\times$$

$$\frac{1+n\times\overline{3dd-1}-6nd}{n\times\overline{3dd-3d}}\ \&c.\times\frac{z^{1+3d}}{c^{n\times\overline{1+3d}}}+\frac{1}{1+4d}\times\frac{1+n\times\overline{4dd-1}-4nd}{4ndd}$$

$$\times\frac{1+n\times\overline{4dd-1}-5nd}{n\times\overline{4dd-d}}\times\frac{1+n\times\overline{4dd-1}-6nd}{n\times\overline{4dd-2d}}\times\frac{1+n\times\overline{4dd-1}-7nd}{n\times\overline{4dd-3d}}$$

$\&c.\times\frac{z^{1+4d}}{c^{n\times\overline{1+4d}}}\mp\&c.$ THEOREM VI.

Which by making $d=3$ gives $z=c^{n}$ multiplying the Series

$$n\times\frac{a}{c}\pm n\times\frac{n-1}{6}\times\frac{n+2}{9}\times\frac{n+5}{12}\times\frac{a^{4}}{c^{4}}+n\times\frac{n-4}{6}\times\frac{n-1}{9}\times\frac{n+2}{12}\times\frac{n+5}{15}$$

$$\times\frac{n+8}{18}\times\frac{n+11}{21}\times\frac{a^{7}}{c^{7}}\pm n\times\frac{n-7}{6}\times\frac{n-4}{9}\times\frac{n-1}{12}\times\frac{n+2}{15}\times\frac{n+5}{18}\times\frac{n+8}{21}\times$$

$$\frac{n+11}{24}\times\frac{n+14}{27}\times\frac{n+17}{30}\times\frac{a^{10}}{c^{10}}+n\times\frac{n-10}{6}\times\frac{n-7}{9}\times\frac{n-4}{12}\times\frac{n-1}{15}\times\frac{n+2}{18}$$

$$\times\frac{n+5}{21}\times\frac{n+8}{24}\times\frac{n+11}{27}\times\frac{n+14}{30}\times\frac{n+17}{33}\times\frac{n+20}{36}\times\frac{n+23}{39}\times\frac{a^{13}}{c^{13}}\pm\ \&c.$$

and $a=\frac{c}{n}$ multiplying the Series $\frac{z}{c^{n}}\mp\frac{n-1}{6n}\times\frac{2n+1}{9n}\times\frac{5n+1}{12n}\times\frac{z^{4}}{c^{4n}}+$

$$\frac{4n-1}{6n}\times\frac{n-1}{9n}\times\frac{2n+1}{12n}\times\frac{5n+1}{15n}\times\frac{8n+1}{18}\times\frac{11n+1}{21n}\times\frac{z^{7}}{c^{7n}}\mp\frac{7n-1}{6n}\times\frac{4n-1}{9n}$$

$$\frac{n-1}{12n}\times\frac{2n+1}{15n}\times\frac{5n+1}{18n}\times\frac{8n+1}{21n}\times\frac{11n+1}{24n}\times\frac{14n+1}{27n}\times\frac{17n+1}{30n}\times\frac{z^{10}}{c^{10n}}+$$

$$\frac{10n-1}{6n}\times\frac{7n-1}{9n}\times\frac{4n-1}{12n}\times\frac{n-1}{15n}\times\frac{2n+1}{18n}\times\frac{5n+1}{21n}\times\frac{8n+1}{24n}\times\frac{11n+1}{27n}\times$$

$\frac{14n+1}{30n}\times\frac{17n+1}{33n}\times\frac{20n+1}{36n}\times\frac{23n+1}{39n}\times\frac{z^{13}}{c^{13n}}\mp$ *&c. Conſectary,* 3.
Which gives a Solution to the following Scale,

$$c^0\ a^1 = z$$

$$4c^3\ a \pm a^4 = z$$

$$7c^6\ a \pm 7c^3a^4 + a^7 = z$$

$$10c^9\ a \pm 25c^6a^4 + 10c^3a^7 \pm a^{10} = z$$

$$13c^{12}a \pm 65c^9a^4 + 52c^6a^7 \pm 13c^3a^{10} + a^{13} = z, \text{ \&c}$$

Again, to make the Reciprocation yet more univerſal, let a^x, $= a^v$; whence (by the Second Section) $n = v + md - d$, and $m = \frac{n+d-v}{d}$. Then in the Value of Q in that Section, by ſubſtituting ſucceſſively v, $v+d$, $v+2d$, $v+3d$, $v+4d$, &c. for x, and reſpectively m, $m-1$, $m-2$, $m-3$, $m-4$, &c. for the n there; we have z^v equal to the *Compoſitum* made up of c^n multiplying the Series $\frac{v+md-d}{v} \times \frac{m+v-2}{v-1} \times \frac{m+v-3}{v-2} \times \frac{m+v-4}{v-3} \times$ &c. $\times \frac{a^v}{c^v} \pm \frac{v+md-d}{v+d} \times \frac{m+v+d-3}{v+d-1} \times \frac{m+v+d-4}{v+d-2} \times \frac{m+v+d-5}{v+d-3}$ &c $\times \frac{a^{v+d}}{c^{v+d}} + \frac{v+md-d}{v+2d} \times \frac{m+v+2d-4}{v+2d-1} \times \frac{m+v+2d-5}{v+2d-2} \times \frac{m+v+2d-6}{v+2d-3}$ &c. $\times \frac{a^{v+2d}}{c^{v+2d}} \pm \frac{v+md-d}{v+3d} \times \frac{m+v+3d-5}{v+3d-1} \times \frac{m+v+3d-6}{v+3d-2} \times \frac{m+v+3d-7}{v+3d-3}$ &c. $\times \frac{a^{v+3d}}{c^{v+3d}} +$ &c. or reſtoring $\frac{n+d-v}{d}$ for m, we have z^v to the *Compoſitum* made up of c^n multiplying the Series $\frac{n}{v} \times \frac{n+d\times\overline{v-1}-v}{d\times\overline{v-1}} \times \frac{n+d\times\overline{v-2}-v}{d\times\overline{v-2}} \times \frac{n+d\times\overline{v-3}-v}{d\times\overline{v-3}}$ &c. $\times$

$\frac{a^v}{c^v} \pm \frac{n}{v+d} \times \frac{n+d\times\overline{v+d-2}-v}{d\times\overline{v+d-1}} \times \frac{n+d\times\overline{v+d-3}-v}{d\times\overline{v+d-2}} \times$ $\frac{n+d\times\overline{v+d-4}-v}{d\times\overline{v+d-3}}$ &c. $\times \frac{a^{v+d}}{c^{v+d}} + \frac{n}{v+2d} \times \frac{n+d\times\overline{v+2d-3}-v}{d\times\overline{v+2d-1}} \times$ $\frac{n+d\times\overline{v+2d-4}-v}{d\times\overline{v+2d-2}} \times \frac{n+d\times\overline{v+2d-5}-v}{d\times\overline{v+2d-3}}$ &c. $\times \frac{a^{v+2d}}{c^{v+2d}} \pm \frac{n}{v+3d} \times$ $\frac{n+d\times\overline{v+3d-4}-v}{d\times\overline{v+3d-1}} \times \frac{n+d\times\overline{v+3d-5}-v}{d\times\overline{v+3d-2}} \times \frac{n+d\times\overline{v+3d-6}-v}{d\times\overline{v+3d-3}}$ &c. $\times \frac{a^{v+3d}}{c^{v+2d}}$ + &c. Conſequently (if we ſubſtitute $\frac{1}{n}$ for n, and $\frac{z}{c^n}$ for $\frac{a}{c}$, as before) we have the *Reſolutum* a^v equal to $\frac{c}{n}$ multiplying the Series $\frac{1}{v} \times \frac{1+n\times\overline{dv-v-d}}{nd\times\overline{v-1}} \times \frac{1+n\times\overline{dv-v-2d}}{nd\times\overline{v-2}} \times$ $\frac{1+n\times\overline{dv-v-3d}}{nd\times\overline{v-3}}$ &c. $\frac{z^v}{c^{nv}} \mp \frac{1}{v+d} \times \frac{1+nd\times\overline{v+d}-n\times\overline{v+2d}}{nd\times\overline{v+d-1}}$ $\times \frac{1+nd\times\overline{v+d}-n\times\overline{v+3d}}{nd\times\overline{v+d-2}} \times \frac{1+nd\times\overline{v+d}-n\times\overline{v+4d}}{nd\times\overline{v+d-3}}$ &c. $\times \frac{z^{v+d}}{c^{n\times\overline{v+d}}}$ $+ \frac{1}{v+2d} \times \frac{1+nd\times\overline{v+2d}-n\times\overline{v+3d}}{nd\times\overline{v+2d-1}} \times \frac{1+nd\times\overline{v+2d}-n\times\overline{v+4d}}{nd\times\overline{v+2d-2}} \times$ $\frac{1+nd\times\overline{v+2d}-n\times\overline{v+5d}}{nd\times\overline{v+2d-3}} \times$ &c. $\times \frac{z^{v+2d}}{c^{n\times\overline{v+2d}}} \mp \frac{1}{v+3d} \times$ $\frac{1+nd\times\overline{v+3d}-n\times\overline{v+4d}}{nd\times\overline{v+3d-1}} \times \frac{1+nd\times\overline{v+3d}-n\times\overline{v+5d}}{nd\times\overline{v+3d-2}} \times$ $\frac{1+nd\times\overline{v+3d}-n\times\overline{v+6d}}{nd\times\overline{v+3d-3}}$ &c. $\times \frac{z^{v+3d}}{c^{n\times\overline{v+3d}}}$ + &c. THEOREM VII.

Hence, if $v = d = 2$, and $z^v = u$, we have $u = \frac{n^2c^2}{4}$ multiplying the Series $\frac{a^2}{c^2} \pm \frac{n-2}{6} \times \frac{n+2}{8} \times \frac{a^4}{c^4} + \frac{n-2}{6} \times \frac{n+2}{8} \times \frac{n-4}{10} \times \frac{n+4}{12} \times \frac{a^6}{c^6} \pm \frac{n-2}{6} \times \frac{n+2}{8} \times \frac{n-4}{10} \times \frac{n+4}{12} \times \frac{n-6}{14} \times \frac{n+6}{16} \times \frac{a^8}{c^8} + \frac{n-2}{6} \times \frac{n+2}{8} \times \frac{n-4}{10} \times \frac{n+4}{12} \times \frac{n-6}{14} \times \frac{n+6}{16} \times \frac{n-8}{18} \times \frac{n+8}{20} \times \frac{a^{10}}{c^{10}} \pm$ &c. $\frac{n-2}{6} \times \frac{n+2}{8} \times \frac{n-4}{10} \times \frac{n+2}{12} \times \frac{n-6}{14} \times \frac{n+6}{16} \times \frac{n-8}{18} \times \frac{n+8}{20} \times \frac{n-10}{22} \times \frac{n+10}{24} \times \frac{a^{12}}{c^{12}} +$ &c. and $a^2 = \frac{c}{4nn}$ multiplying the Series $\frac{u}{c^{2n}} \mp \frac{2n-1}{6n} \times \frac{2n+1}{8n} \times \frac{u^2}{c^{4n}} + \frac{2n-1}{6n} \times \frac{2n+1}{8n} \times \frac{4n-1}{10n} \times \frac{4n+1}{12n} \times \frac{u^3}{c^{6n}} \mp \frac{2n-1}{6n} \times \frac{2n+1}{8n} \times \frac{4n-1}{10n} \times \frac{4n+1}{12n} \times \frac{6n-1}{14n} \times \frac{6n+1}{16n} \times \frac{u^4}{c^{8n}} + \frac{2n-1}{6n} \times \frac{2n+1}{8n} \times \frac{4n-1}{10n} \times \frac{4n+1}{12n} \times \frac{6n-1}{14n} \times \frac{6n+1}{16n} \times \frac{8n-1}{18n} \times \frac{8n+1}{20n} \times \frac{u^5}{c^{10n}} \mp \frac{2n-1}{6n} \times \frac{2n+1}{8n} \times \frac{4n-1}{10n} \times \frac{4n+1}{12n} \times \frac{6n-1}{14n} \times \frac{6n+1}{16n} \times \frac{8n-1}{18n} \times \frac{8n+1}{20n} \times \frac{10n-1}{22n} \times \frac{10n+1}{24n} \times \frac{u^6}{c^{12n}} +$ &c. *Conſectary* 4. Which gives a Solution to all Equations of the following Scale.

$c^0 \ a^2 \quad = z^2 = u$

$4c^2 \ a^2 \pm a^4 \quad = z^2 = u$

$9c^4 \ a^2 \pm 6c^2a^4 + a^6 \quad = z^2 = u$

$16c^6 \ a^2 \pm 20c^4a^4 + 8c^2a^6 \pm a^8 \quad = z^2 = u$

$25c^8 \ a^2 \pm 50c^6a^4 + 35c^4a^6 \pm 10c^2a^8 + a^{10} \quad = z^2 = u$

$36c^{10}a^2 \pm 105c^8a^4 + 112c^6a^6 \pm 54c^4a^8 + 12c^2a^{10} \pm a^{12} = z^2 = u$&c.

And if $c=1$, the ſame Scale will be expreſſed by

$$\begin{array}{ll} a^2 & =u \\ 4a^2 \pm a^4 & =u \\ 9a^2 \pm 6a^4 + a^6 & =u \\ 16a^2 \pm 20a^4 + 8a^6 \pm a^8 & =u \\ 25a^2 \pm 50a^4 + 35a^6 \pm 10a^8 + a^{10} & =u \\ 36a^2 \pm 105a^4 + 112a^6 \pm 54a^8 + 12a^{10} \pm a^{12} & =u, \text{ \&c.} \end{array}$$

If the ſquare Roots of the 1ſt, 3d, 5th, 7th, &c. Steps be extracted, we ſhall have a new Scale expreſſed in the ſame Terms with the Scale in the ſecond Conſectary above, with each Tearm equal to $u^{\frac{1}{2}}$. But becauſe the n here is only equal to $\frac{n}{2}$ there; therefore by ſubſtituting $\frac{n}{2}$ for n in the ſecond Conſectary, and 1 for c, we have $2 \times u^{\frac{1}{2}} = 2z = x = na \pm n \times \frac{n-2}{8} \times \frac{n+2}{12} a^3 + n \times \frac{n-2}{8} \times \frac{n+2}{12} \times \frac{n-6}{16} \times \frac{n+6}{20} a^5 \pm n \times \frac{n-2}{8} \times \frac{n+2}{12} \times \frac{n-6}{16} \times \frac{n+6}{20} \times \frac{n-10}{24} \times \frac{n+10}{28} a^7 + n \times \frac{n-2}{8} \times \frac{n+2}{12} \times \frac{n-6}{16} \times \frac{n+6}{20} \times \frac{n-10}{24} \times \frac{n+10}{28} \times \frac{n-14}{32} \times \frac{n+14}{36} a^9 \pm$ &c. Alſo by ſubſtituting 1 for c, $\frac{n}{2}$ for n, and $\frac{x}{2}$ for z in the *Reſolutum* of the ſaid Conſectary, we have $a = \frac{1}{n}$ multiplying the Series $x \mp \frac{n-2}{8n} \times \frac{n+2}{12n} x^3 + \frac{n-2}{8n} \times \frac{n+2}{12n} \times \frac{3n-2}{16n} \times \frac{3n+2}{20n} x^5 \mp \frac{n-2}{8n} \times \frac{n+2}{12n} \times \frac{3n-2}{16n} \times \frac{3n+2}{20n} \times \frac{5n-2}{24n} \times \frac{5n+2}{28n}$

$\frac{5n+2}{28n}x^7+\frac{n-2}{8n}\times\frac{n+2}{12n}\times\frac{3n-2}{16n}\times\frac{3n+2}{20n}\times\frac{5n-2}{24n}\times\frac{5n+2}{28n}\times\frac{7n-2}{32n}\times\frac{7n+2}{36n}x^9\mp$ *&c.* *Confectary* 5.

Again, if the 2d, 4th, 6th, 8th, *&c.* Steps of the laſt Scale be each added to and ſubtracted from 4, we have $4\pm4a^2+a^4=4\pm u$, $4\pm16a^2+20a^4\pm8a^6+a^8=4\pm u$, $4\pm36a^2+105a^4\pm112a^6+54a^8\pm12a^{10}+a^{12}=4\pm u$, *&c.* Conſequently the Square Roots of each will give $2\pm a^2=\overline{4\pm u}^{\frac{1}{2}}$, $2\pm4a^2+a^4=\overline{4\pm u}^{\frac{1}{2}}$, $2\pm9a^2+6a^4\pm a^6=\overline{4\pm u}^{\frac{1}{2}}$, $2\pm16a^2+20a^4\pm8a^6+a^8=\overline{4\pm u}^{\frac{1}{2}}$, *&c.* Whence, if the 2 in each be tranſpoſed to the contrary Side, we ſhall have a Scale expreſſed in the ſame Terms with that of the fourth Confectary, with each Step equal to $\overline{4+u}^{\frac{1}{2}}-2$ or $2-\overline{4-2}^{\frac{1}{2}}$, ſubſtituting 1 for c, and $\frac{n}{2}$ for n in that Confectary; becauſe each Step here is the Square or $\frac{1}{2}$ Root of each Step there.

And by the ſame Procedure from the laſt Scale, as from that of the fourth Confectary, we may have a third Scale expreſſed ſtill in the ſame Terms with each Step equal to $\overline{\overline{4+u}^{\frac{1}{2}}+2}^{\frac{1}{2}}-2$ or $2\overline{-\overline{4-u}^{\frac{1}{2}}+2}^{\frac{1}{2}}$ having $\frac{n}{4}$ for n in the Serieſes of the fourth Confectary. And ſo on to $\frac{n}{8}$, $\frac{n}{16}$, $\frac{n}{32}$, *&c.* for n there.

Now it is apparent, (that proceeding as was done from the fourth to the fifth Confectary) by extracting the Square Roots of the firſt, third, fifth, *&c.* Steps of the Scale, whoſe Terms are each equal to $\overline{4+u}^{\frac{1}{2}}-2$ or $2-\overline{4-u}^{\frac{1}{2}}$ a Scale will again be produced, and expreſſed in the ſame Terms with the Scale again of the ſecond Confectary, having $1=c$ and each ſtep equal to $\overline{\overline{4+u}^{\frac{1}{2}}-2}^{\frac{1}{2}}$ or $\overline{2-\overline{4-u}^{\frac{1}{2}}}^{\frac{1}{2}}$. But becauſe the n here is only equal $\frac{n}{4}$; therefore by ſubſtituting $\frac{n}{4}$ for n in the *Compoſitum* of the ſecond Confectary, and 1 for c, we have $4\times\overline{\overline{4+u}^{\frac{1}{2}}-2}^{\frac{n}{2}}=4z=x$ or $4\times\overline{2-\overline{4-u}^{\frac{1}{2}}}^{\frac{1}{2}}=4z=x$ $x=na\pm n\times\frac{n-4}{16}\times\frac{n+4}{24}a^3+n\times\frac{n-4}{16}\times\frac{n+4}{24}\times\frac{n-12}{32}\times\frac{n+12}{40}a^5\pm n\times\frac{n-4}{16}\times\frac{n+4}{24}\times\frac{n-12}{32}\times\frac{n+12}{40}\times\frac{n-20}{48}\times\frac{n+20}{56}a^7+$ &c. And by ſubſtituting 1 for c, $\frac{n}{4}$ for n, and $\frac{x}{4}$ for z in the *Reſolutum* there, we have $a=\frac{1}{n}$ multiplying the Series $x-\frac{n-4}{16n}\times\frac{n+4}{24n}x^3+\frac{n-4}{16n}\times\frac{n+4}{24n}\times\frac{3n-4}{32n}\times\frac{3n+4}{40n}x^5\mp\frac{n-4}{16n}\times\frac{n+4}{24n}\times\frac{3n-4}{32n}\times\frac{3n+4}{40n}\times\frac{5n-4}{48n}\times\frac{5n+4}{56n}x^7+$ &c. *Confectary* 6.

And univerſally how high ſoever the Value of each Step in the Scale of the fourth confectary be carried (as from u to $\pm\overline{4\pm u^{\frac{1}{2}}}\mp2$ to $\pm\overline{\overline{4\pm u}^{\frac{1}{2}}+2}^{\frac{1}{2}}\mp2$ to &c.) by extracting the ſquare Roots of the firſt, third, fifth, &c. Terms, a new Scale will be ſtill formed and expreſſed in the ſame Terms with the Scale of the ſecond Confectary with 2^1, 2^2; 2^3, 2^4, 2^5, &c. and univerſally d to divide the n in the Serieſes of that Confectary. So that univerſally, putting 1 for c, $\frac{n}{d}$ for n, and $\frac{x}{d}$ for z in thoſe Serieſes, we have $x=na\pm n\times\frac{n-d}{4d}\times\frac{n+d}{6d}a^3+n\times\frac{n-d}{4d}\times\frac{n+d}{6d}\times\frac{n-3d}{8d}\times\frac{n+3d}{10d}a^5\pm n\times\frac{n-d}{4d}\times\frac{n+d}{6d}\times\frac{n-3d}{8d}\times\frac{n+3d}{10d}\times\frac{n-5d}{12d}\times\frac{n+5d}{14d}$

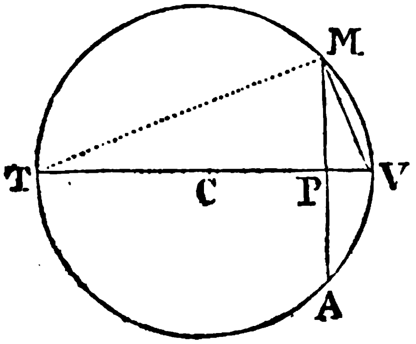
$a^7\pm$

$a^7 +$ &c. And $a = \frac{1}{n}$ multiplying the Series $x \mp \frac{n-d}{4dn} \times \frac{n+d}{6dn} \cdot x^3 + \frac{n-d}{4dn} \times \frac{n+d}{6dn} \times \frac{3n-d}{8dn} \times \frac{3n+d}{10dn} x^5 \mp \frac{n-d}{4dn} \times \frac{n+d}{6nd} \times \frac{3n-d}{8dn} \times \frac{3n+d}{10dn} \times \frac{5n-d}{12dn} \times \frac{5n+d}{14dn} x^7 +$ &c. THEOREM VIII.

SECT. V.

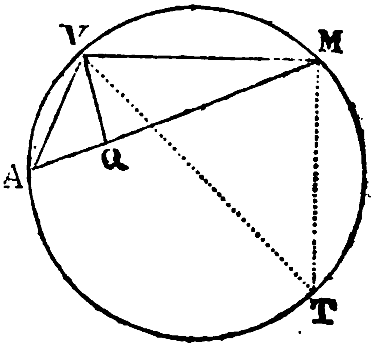
An Application of the foregoing Doctrine to the angular Sections of a Circle.

IN every Circle make the Radius TC = CV = c, the Sine or Semiordinate MP = $\frac{1}{2}$Z = s; the verſed Sine or Abſciſſa VP = v, and the Chord VM (of half the Arch AM) = z. Then by Reaſon of the ſimilar right angled Triangles VMT, PVM, and PMT; we have VP : PM = PM : PT, or $v : \frac{1}{2}Z = \frac{1}{2}Z : 2c - v$. Whence $2cv - vv = \frac{1}{4}ZZ = ss$, *Conſectary* 1. VP : VM = MP : MT, or $v : z = \frac{1}{2}Z : \frac{Zz}{2v} = \frac{sz}{v} =$ MT, *Conſectary* 2. MT : VT = MP : VM, or $\frac{sz}{v} : 2c = s : z$; whence $z^2 = 2cv$, *Conſectary* 3. And ſubſtituting z^2 for $2cv$ in the Equation $2cv - v^2 = ss$, we have $zz - vv = ss$, *Conſectary* 4, the Property of all right-angled Triangles. Whence $z^2 - \frac{1}{4}ZZ = vv$; and ſince (Conſect. 3.) $z^4 = 4c^2v^2$; therefore $4c^2z^2 - c^2Z^2 = z^4$, or $4c^2z^2 - z^4 = c^2Z^2$, *Conſectary* 5. Which laſt being the conſtant Relation between every Chord MA = Z, and the Chord MV = z of half the Arch AVM; therefore if we put e = the Chord of $\frac{1}{4}$ Part, u = the Chord of $\frac{1}{8}$, y = the



$=$ the Chord of $\frac{1}{16}$, *&c.* of the ſame Arch A V M; we ſhall the ſame Way have $c^2z^2 = 4c^2e^2 - e^4$, $c^2e^2 = 4c^2u^2 - u^4$, $c^2u^2 = 4c^2y^2 - y^4$, *&c.* Conſequently, if we ſubſtitute every latter in the Place of the foamer, putting $c =$ Unity, we have $ZZ = 4z^2 - z^4 = 16e^2 - 20e^4 + 8e^6 - e^8 = 64u^2 - 336u^4 + 672u^6 - 660u^8 + 352u^{10} - 104u^{12} + 16u^{14} - u^{16} =$ *&c.* *Conſectary* 6. And (Conſect. 3.) $2v = z^2 = 4e^2 - e^4 = 16u^2 - 20u^4 + 8u^6 - u^8 = 64y^2 - 336y^4 + 672y^6 - 660y^8 + 352y^{10} - 104y^{12} + 16y^{14} - y^{16} =$ *&c.* *Conſectary* 7.

Let the Triangle A V M be inſcribed within a Circle, and from V let fall V Q perpendicular to A M, drawing the Diameter $VT = 2c = 2$; and join MT. Then the Angle V M T being right (as ſtanding within a Semicirle), and the Angle V T M being equal to V A M (as both ſtanding upon the ſame Chord V M) the Angle VTM and VAQ muſt be ſimilar. Conſequently V T : VM, or $\overline{VT}^2 : \overline{VM}^2 = \overline{VA}^2 : \overline{VQ}^2$ *Conſectary* 8.



If then the Arch A VM be divided into three equal Parts, whoſe Chords are each equal to a; and A V contains one of thoſe Parts, and V M two (the Chord of the whole being $AM = Z$); we have (Conſect. 5.) $\overline{VM}^2 = 4a^2 - a^4$, and (Conſect. 8.) $\overline{VQ}^2 = a^4 - \frac{1}{4}a^6$. Therefore (becauſe $\overline{AV}^2 - \overline{VQ}^2 = \overline{a - \frac{1}{2}aaa}^2 = \overline{AQ}^2$, and $\overline{2a - \frac{1}{2}aaa}^2 = \overline{VM}^2 - \overline{VQ}^2$) we have $AQ + QM = Z = 3a - a^3$ for the Triſection of an Arch or Angle, *Conſectary* 9. Conſequently, if e equal the Chord of $\frac{1}{9}$ Part of the ſame Arch (whence $a = 3e - e^3$); we have $Z = 3a - a^3 = 9e - 30e^3 + 27e^5 - 9e^7 + e^9 =$ *&c.* *Conſectary* 10.

If the Arch A V M be divided into four equal Parts, whoſe Chords are each equal to e; and A V and V M contains each two of thoſe Parts; whence $VQ = v$ the verſed Sine or Abſciſſa of the whole Arch: Then (Conſect. 7.) $2v = 4e^2 - e^4$.

If the Arch AVM be divided into five equal Parts, whoſe Chords are equal to a; and AV contains two of thoſe Parts, and VM three (the Chord of the whole being $AM = Z$): Then (Conſect. 6.) $\overline{AV}^2 = 4a^2 - a^4$, and (Conſect. 9) $\overline{VM}^2 = \overline{3a - aaa}^2 = 9a^2 - 6a^4 + a^6$, and (Conſect. 8.) $\overline{VQ}^2 = \frac{\overline{VM}^2 \times \overline{AV}^2}{\overline{VT}^2} = 9a^4 - \frac{33}{4}a^6 + \frac{5}{2}a^8 - \frac{1}{4}a^{10}$. Hence (becauſe $\overline{AV}^2 - \overline{VQ}^2 = \overline{2a - \frac{5}{2}a^3 + \frac{1}{2}a^5}^2 = \overline{AQ}^2$, and $\overline{VM}^2 - \overline{VQ}^2 = \overline{3a - \frac{5}{2}a^3 + \frac{1}{2}a^5}^2 = \overline{QM}^2$) we have $AQ + QM = AM$ or $5a - 5a^3 + a^5 = Z$, *Conſectary* 11.

If the Arch AVM be divided into ſix equal Parts, whoſe Chords are each equal to a; and AV and VM contains each three of thoſe Parts; whence $VQ = v$, and (Conſect. 9.) $\overline{AV}^2 = \overline{VM}^2 = \overline{3a - aaa}^2 = 9a^2 - 6a^4 + a^6$: Then (Conſect. 8.) $9a^6 - 6a^4 + a^6 = 2v$. *Conſectary* 12.

If the Arch AVE be divided into ſeven equal Parts, whoſe Chords are each equal to a, and AV contains two of thoſe Parts and VM five (the Chord of the whole Arch being $AM = Z$); then (Conſect. 5.) $\overline{AV}^2 = 4a^2 - a^4$, and (Conſect. 11.) $\overline{VM}^2 = \overline{5a - 5a^3 + a^5}^2 = 25a^2 - 50a^4 + 35a^6 - 10a^8 + a^{10}$, and (Conſect. 8.) $\overline{VQ}^2 = 25a^4 - \frac{225}{4}a^6 + \frac{190}{4}a^8 - \frac{75}{4}a^{10} + \frac{14}{4}a^{12} - \frac{1}{4}a^{14}$. Hence (becauſe $\overline{AV}^2 - \overline{VQ}^2 = \overline{AQ}^2 = \overline{2a - \frac{13}{2}a^3 + \frac{7}{2}a^5 - \frac{1}{2}a^7}^2$, and $\overline{VM}^2 - \overline{VQ}^2 = \overline{5a - \frac{15}{2}a^3 + \frac{7}{2}a^5 - \frac{1}{2}a^7}^2 = \overline{QM}^2$), we have $AQ + QM = AM = Z = 7a - 14a^3 + 7a^5 - a^7$, *Conſectary* 13.

And thus proceeding, with $1 = c$, and putting univerſally $v =$ the Abſciſſa, $z = Z =$ the given Chord, and $a =$ the Chord required; we ſhall find all the odd Diviſions of a circular Arch, *viz.* into 3, 5, 7, 9, 11, &c. equal Parts to proceed exactly according to the Scale of the ſecond Confectary in the foregoing Section, having $c = 1$, as follows.

$a = z$

$3a - a^3 = z$ (by Confect. 9 of this Sect.)

$5a - 5a^3 + a^5 = z$ (Confect. 11.)

$7a - 14a^3 + 7a^5 - a^7 = z$ (Confect. 13.)

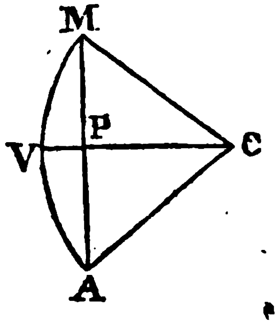
$9a - 30a^3 + 27a^5 - 9a^7 + a^9 = z$ (Confect. 10), *&c.*

$11a - 55a^3 + 77a^5 - 44a^7 + 11a^9 - a^{11} = z$, *&c.*

Conſequently, from the ſame Confectary there, whatever n be we have the *Compoſitum* $z = na - n \times \frac{n-1}{4} \times \frac{n+1}{6} a^3 + n \times \frac{n-1}{4} \times \frac{n+1}{6} \times \frac{n-3}{8} \times \frac{n+3}{10} a^5 - n \times \frac{n-1}{4} \times \frac{n+1}{6} \times \frac{n-3}{8} \times \frac{n+3}{10} \times \frac{n-5}{12} \times \frac{n+5}{14} a^7 + n \times \frac{n-1}{4} \times \frac{n+1}{6} \times \frac{n-3}{8} \times \frac{n+3}{10} \times \frac{n-5}{12} \times \frac{n+5}{14} \times \frac{n-7}{16} \times \frac{n+7}{18} a^9 - n \times \frac{n-1}{4} \times \frac{n+1}{6} \times \frac{n-3}{8} \times \frac{n+3}{10} \times \frac{n-5}{12} \times \frac{n+5}{14} \times \frac{n-7}{16} \times \frac{n+7}{18} \times \frac{n-9}{20} \times \frac{n+9}{22} a^{11} +$ &c. And in the *Reſolutum* $a = \frac{1}{n}$ multiplying the Series $z + \frac{n-1}{4n} \times \frac{n+1}{6n} z^3 + \frac{n-1}{4n} \times \frac{n+1}{6n} \times \frac{3n-1}{8n} \times \frac{3n+1}{10n} z^5 + \frac{n-1}{4n} \times \frac{n+1}{6n} \times \frac{3n-1}{8n} \times \frac{3n+1}{10n} \times \frac{5n-1}{12n} \times \frac{5n+1}{14n} z^7 + \frac{n-1}{4n} \times \frac{n+1}{6n} \times \frac{3n-1}{8n} \times \frac{3n+1}{10n} \times \frac{5n-1}{12n} \times \frac{5n+1}{14n} \times \frac{7n-1}{16n} \times \frac{7n+1}{18n} z^9 + \frac{n-1}{4n} \times \frac{n+1}{6n} \times$

$\frac{3n-1}{8n}\times\frac{3n+1}{10n}\times\frac{5n-1}{12n}\times\frac{5n+1}{14n}\times\frac{7n-1}{16n}\times\frac{7n+1}{18n}\times\frac{9n-1}{20n}\times\frac{9n+1}{22n}z^{11}+$ &c. *Conſectary* 14.

But if we chuſe to work by the Sines, or half Chords, put $s=\frac{1}{2}z$, and $y=$ the Sine of half the Arch, whoſe Chord is a, whence $2y=a$, and $2s=z$; and as na in the laſt Conſectary equals the Sum of all the Chords of the whole Arch AVM, ſo is ny here equal to the Sum of all the Sines or Semiordinates of the Half-arch VM. Therefore in the laſt Serieſes ſubſtituting $2y$ for a and $2s$ for z, dividing the Whole of each Equation by 2; we have $s=ny-n\times\frac{n-1}{2}\times\frac{n+1}{3}y^3+n\times\frac{n-1}{2}\times\frac{n+1}{3}\times\frac{n-3}{4}\times\frac{n+3}{5}y^5-n\times\frac{n-1}{2}\times\frac{n+1}{3}\times\frac{n-3}{4}\times\frac{n+3}{5}\times\frac{n-5}{6}\times\frac{n+5}{7}y^7+n\times\frac{n-1}{2}\times\frac{n+1}{3}\times\frac{n-3}{4}\times\frac{n+3}{5}\times\frac{n-5}{6}\times\frac{n+5}{7}\times\frac{n-7}{8}\times\frac{n+7}{9}y^9-$ *&c.* and $y=\frac{1}{n}$ multiplying the Series $s+\frac{n-1}{2n}\times\frac{n+1}{3n}s^3+\frac{n-1}{2n}\times\frac{n+1}{3n}\times\frac{3n-1}{4n}\times\frac{3n+1}{5n}s^5+\frac{n-1}{2n}\times\frac{n+1}{3n}\times\frac{3n-1}{4n}\times\frac{3n+1}{5n}\times\frac{5n-1}{6n}\times\frac{5n+1}{7n}s^7+\frac{n-1}{2n}\times\frac{n+1}{3n}\times\frac{3n-1}{4n}\times\frac{3n+1}{5n}\times\frac{5n-1}{6n}\times\frac{5n+1}{7n}\times\frac{7n-1}{8n}\times\frac{7n+1}{9n}s^9+$ &c. *Conſectary* 15. *



Again, it appears from the Procedure of the thirteen firſt Conſectaries of this Section, that all the even Diviſions of a circular Arch, *viz.* into 2, 4, 6, 8, 10, *&c.* equal Parts (repreſenting all Things, as before) do proceed exactly according to the Scale of the fourth Conſectary of the laſt Section, having $c=r=1$, and u there equal to $2v$ here; as follows

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* *Again, the Learner, if he pleaſes, may ſtep here and proceed to the following Section, for the ſame Reaſons as in the laſt.*

a^2

$$a^2 = 2v \text{ (Confect 3.)}$$

$$4a^2 - a^4 = 2v \text{ (Confect 7.)}$$

$$9a^2 - 6a^4 + a^6 = 2v \text{ (Confect 12.)}$$

$$16a^2 - 20a^4 + 8a^6 - a^8 = 2v \text{ (Confect 7.)}$$

$$25a^2 - 50a^4 + 35a^6 - 10a^8 + a^{10} = 2v$$

$$36a^2 - 105a^4 + 112a^6 - 54a^8 + 12a^{10} - a^{12} = 2v, \text{ \&c.}$$

Conſequently, from the ſame fourth Confectary in the laſt Section, whatever n be, we have in the *Compoſitum*, $2v =$ to $\frac{nn}{4}$ multiplying the Series $a^2 - \frac{n-2}{6} \times \frac{n+2}{8} a^4 + \frac{n-2}{6} \times \frac{n+2}{8} \times \frac{n-4}{10} \times \frac{n+4}{12} a^6 - \frac{n-2}{6} \times \frac{n+2}{8} \times \frac{n-4}{10} \times \frac{n+4}{12} \times \frac{n-6}{14} \times \frac{n+6}{16} a^8 + \frac{n-2}{6} \times \frac{n+2}{8} \times \frac{n-4}{10} \times \frac{n+4}{12} \times \frac{n-6}{14} \times \frac{n+6}{16} \times \frac{n-8}{18} \times \frac{n+8}{20} a^{10} - \frac{n-2}{6} \times \frac{n+2}{8} \times \frac{n-4}{10} \times \frac{n+4}{12} \times \frac{n-6}{14} \times \frac{n+6}{16} \times \frac{n-8}{18} \times \frac{n+8}{20} \times \frac{n-10}{22} \times \frac{n+10}{24} a^{12} +$ &c. and the *Reſolutum* $a = \frac{1}{4nn}$ multiplying the Series $u + \frac{2n-1}{6n} \times \frac{2n+1}{8n} u^2 + \frac{2n-1}{6n} \times \frac{2n+1}{8n} \times \frac{4n-1}{10n} \times \frac{4n+1}{12n} u^3 + \frac{2n-1}{6n} \times \frac{2n+1}{8n} \times \frac{4n-1}{10n} \times \frac{4n+1}{12n} \times \frac{6n-1}{14n} \times \frac{6n+1}{16n} u^4 + \frac{2n-1}{6n} \times \frac{2n+1}{8n} \times \frac{4n-1}{10n} \times \frac{4n+1}{12n} \times \frac{6n-1}{14n} \times \frac{6n+1}{16n} \times \frac{8n-1}{18n} \times \frac{8n+1}{20n} u^5 + \frac{2n-1}{6n} \times \frac{2n+1}{8n} \times \frac{4n-1}{10n} \times \frac{4n+1}{12n} \times \frac{6n-1}{14n} \times \frac{6n+1}{16n} \times \frac{8n-1}{18n} \times \frac{8n+1}{20n} \times \frac{10n-1}{22n} \times \frac{10n+1}{24n} u^6 +$ &c. *Confectary* 16.

If therefore the ſquare Roots of the firſt, third, fifth, ſeventh, &c. Steps of the laſt Scale here be extracted, it is evident they will give a new Scale of Equations for finding the Chord a of

$\frac{1}{2}, \frac{1}{6}, \frac{1}{10}, \frac{1}{14}, \frac{1}{18}$, *&c.* Part of the given Angle expreſſed in the ſame Terms with the Scale of the fourteenth Conſectary of this Section; each Step being equal to $\overline{2v}^{\frac{1}{2}} = u^{\frac{1}{2}} = \frac{1}{2}x$. Conſequently (by Conſect. 5. of the laſt Section), we have here $x = na - n\times\frac{n-2}{8}\times\frac{n+2}{12}a^3 + n\times\frac{n-2}{8}\times\frac{n+2}{12}\times\frac{n-6}{16}\times\frac{n+6}{20}a^5 - n\times\frac{n-2}{8}\times\frac{n+2}{12}\times\frac{n-6}{16}\times\frac{n+6}{20}\times\frac{n-10}{24}\times\frac{n+10}{28}a^7 + n\times\frac{n-2}{8}\times\frac{n+2}{12}\times\frac{n-6}{16}\times\frac{n+6}{20}\times\frac{n-10}{24}\times\frac{n+10}{28}\times\frac{n-14}{32}\times\frac{n+14}{36}a^9 -$ *&c.* and $a = \frac{1}{n}$ multiplying the Series $x + \frac{n-2}{8n}\times\frac{n+2}{12n}x^3 + \frac{n-2}{8n}\times\frac{n+2}{12n}\times\frac{3n-2}{16n}\times\frac{3n+2}{20n}x^5 + \frac{n-2}{8n}\times\frac{n+2}{12n}\times\frac{3n-2}{16n}\times\frac{3n+2}{20n}\times\frac{5n-2}{24n}\times\frac{5n+2}{28n}x^7 + \frac{n-2}{8n}\times\frac{n+2}{12n}\times\frac{3n-2}{16n}\times\frac{3n+2}{20n}\times\frac{5n-2}{24n}\times\frac{5n+2}{28n}\times\frac{7n-2}{32n}\times\frac{7n+2}{36n}x^9 +$ &c. *Conſectary* 17.

Again, if we proceed with the Scale of the 16th Conſectary here, according to the Directions for the ſixth Conſectary of the laſt Section, becauſe $2v$ here equals u there; we have $4\times\overline{2-4-\overline{2v}^{\frac{1}{2}}}^{\frac{1}{2}} = x = na - n\times\frac{n-4}{16}\times\frac{n+4}{24}a^3 + n\times\frac{n-4}{16}\times\frac{n+4}{24}\times\frac{n-12}{32}\times\frac{n+12}{40}a^5 - n\times\frac{n-4}{16}\times\frac{n+4}{24}\times\frac{n-12}{32}\times\frac{n+12}{40}\times\frac{n-20}{48}\times\frac{n+20}{56}a^7 + \frac{n-4}{16}\times\frac{n+4}{24}\times\frac{n-12}{32}\times\frac{n+12}{40}\times\frac{n-20}{48}\times\frac{n+20}{56}\times\frac{n-28}{64}\times\frac{n+28}{72}a^9 -$ *&c.* and $a = \frac{1}{n}$ multiplying the Series $x + \frac{n-4}{16n}\times\frac{n+4}{24n}x^3 + \frac{n-4}{10n}\times\frac{n+4}{24n}\times\frac{3n-4}{32n}\times\frac{3n+4}{40n}x^5 + \frac{n-4}{16n}\times\frac{n+4}{24n}\times\frac{nn-4}{32n}\times\frac{3n+4}{40n}\times\frac{5n-4}{48n}\times\frac{5n+4}{56n}x^7 + \frac{n-4}{16n}\times\frac{n+4}{24n}\times\frac{3n-4}{32n}\times\frac{3n+4}{40n}\times\frac{5n-4}{48n}\times\frac{5n+4}{56n}\times\frac{7n-4}{64n}\times\frac{7n+4}{72n}x^9 +$ &c. *Conſectary* 18. Which gives the Value of the Chord a to the 4th, 12th, 20th, 28th, 36th, 44th, *&c.* Parts of the given Angle.

And thus, according to the Directions for the 8th Theorem in the laſt Section, we have univerſally $x = na - n \times \frac{n-d}{4d} \times \frac{n+d}{6d} a^3 + \frac{n-d}{4d} \times \frac{n+d}{6n} \times \frac{n-3d}{8d} \times \frac{n+3d}{10d} a^5 - \frac{n-d}{4d} \times \frac{n+d}{6d} \times \frac{n-3d}{8d} \times \frac{n+3d}{10d} \times \frac{n-5d}{12d} \times \frac{n+5d}{14d} x^7 + \&c.$ And $a = \frac{1}{n}$ multiplying the Series $x + \frac{n-d}{4dn} \times \frac{n+d}{6dn} x^3 + \frac{n-d}{4dn} \times \frac{n+d}{6dn} \times \frac{3n-d}{8dn} \times \frac{3n+d}{10dn} \times \frac{5n-d}{12dn} \times \frac{5n+d}{14dn} + \&c.$ THEOREM IX. That is,

If $d = 8$ then for finding the Chord a of the 8th, 24th, 40th, 56th, 72th, 88th, *&c.* Parts of the given Angle; we have $x = 8$

$$\times \overline{2 - \overline{2 + \overline{4 - 2v}^{\frac{1}{2}}}^{\frac{1}{2}}}^{\frac{1}{2}}$$ *Conſectary* 19.

If $d = 16$; then for finding the Chord a of the 16th, 48th, 112th, 144th, 176th, &c. Parts of the given Angle, we have

$$x = 16 \times \overline{2 - \overline{2 + \overline{2 + \overline{4 - 2v}^{\frac{1}{2}}}^{\frac{1}{2}}}^{\frac{1}{2}}}^{\frac{1}{2}}$$

Conſectary 20. &c. *ad infinitum.*

SECT. VI.

The Menſuration of the Arch of a Circle.

THE Menſuration of the Arch of a Circle, and every other Algebraic Curve, is founded upon the following Principles.

I. Space, and conſequently Quantity and Number, may be conſidered as encreaſed, *ad infinitum.*

II. If any finite Quantity e, z, or x be ſuppoſed to be divided by n put to repreſent Number univerſally encreaſed, *ad infinitum*; then it is plain, that the Quotients $\frac{e}{n}, \frac{z}{n}, \frac{x}{n}$, can be no Quantities, but mere Negations of all Poſſibility for their reſpective Dividends e, z, or x, to be divided further. Conſequently, if the Dividend be a Line, the Quotient muſt be a mere Geometrical Point; if the Dividend be a Superficies, the Quotient muſt be a Line, or indiviſible Breadth; and if the Dividend be a Solid, the Quotient muſt be a Superficies, or indiviſible Thickneſs.

III. Under this Suppoſition, therefore the Quotients $\frac{e}{n}, \frac{z}{n}, \frac{x}{n}$, may be aptly called *Indiviſibles*; whoſe reſpective *Diviſibles* are the Dividends e, z, x.

IV. If n be put univerſally to repreſent Number encreaſed *ad infinitum*; then every aſſignable Number, how great ſoever, muſt be as nothing with Reſpect to it. Conſequently the Infinite n can neither be encreaſed by the Addition, nor decreaſed by the Subtraction of any aſſignable Number whatever.

V. It is evident that in all the Diviſions of a given Angle ACM = 2 VCM, or VCM (ſee the laſt Figure of the laſt Section), the more equal Angles the ſaid given Angle is divided into; ſtill the nearer muſt the Sum n a of all the Chords of the Angle A C M approach to an Equality with the Length of the Arch A V M; and the Sum n y of all the Sines of the Angle V C M $= \frac{1}{2}$ A C M approach to an Equality with the Length of the Arch V M $= \frac{1}{2}$ A V M. If therefore the Exponent n of the 14th and 15th Conſectaries (which is there always equal to the Number of Parts the Angle is ſuppoſed to be divided into) be put to repreſent Number univerſally encreaſed *ad infinitum*, and conſequently the Chords a, and the Sines y $= \frac{1}{2}$ a are infinitely little; it follows, that n a muſt be the very ſame with the Arch A V M, and n y the very ſame with the Arch V M. Becauſe in that Caſe, the equal Differences, between theſe infinitely little Chords or Sines and their reſpective equal Arches, muſt be as infinitely leſs than thoſe infinitely little Chords a or Sines y and their reſpective Arches, as thoſe Chords, Sines or Arches themſelves are leſs than the Arch propoſed to be divided. For if we put e univerſally to repreſent the Length of every Arch to be divided into any n Number of equal Parts, one of which Parts is x, whoſe correſpondent Chord is a; it is plain that n x muſt be always leſſer than n a + a, and conſequently n x — n a is leſſer than a. But if a be indiviſible, then the Difference between n x and n a muſt be alſo indiviſible, as being leſſer than a. That is n x = n a. And if n x — n a be leſſer than an indiviſible, then (dividing by the Infinite n) we have x — a more than infinitely leſſer than that Indiviſible. And the ſame for the Sines.

Hence if in the fourteenth Conſectary of the laſt Section, the n there be put to repreſent Number encreaſed *ad infinitum*; it is evident (by the fourth Principle here laid down) that all the finite Numbers —1, +1, —2, +2, —3, +3, —4, +4, &c. in the Numerators of the *Unciæ* muſt vaniſh as nothing. And conſequently the *Compoſitum* there becomes $AM = z = na - \frac{1}{4} \times \frac{1}{6} n^3 a^3 + \frac{1}{4} \times \frac{1}{6} \times \frac{1}{8} \times \frac{1}{10} n^5 a^5 - \frac{1}{4} \times \frac{1}{6} \times \frac{1}{8} \times \frac{1}{10} \times \frac{1}{12} \times \frac{1}{14} n^7 a^7 +$ &c. Or (becauſe n a = e = the Arch A V M) $AM = z = e + \frac{1}{4} \times \frac{1}{6} e^3 + \frac{1}{4} \times \frac{1}{6} \times \frac{1}{8} \times \frac{1}{10} e^5 - \frac{1}{4} \times \frac{1}{6} \times \frac{1}{8} \times \frac{1}{12} \times \frac{1}{14} e^7 +$ &c. And for the ſame Reaſon the *Reſolutum* there becomes (by multi-

tiplying

plying all by n) $AVM = na = e = z + \frac{n}{5n} \times \frac{n}{6n} z^3 + \frac{n}{4n} \times \frac{n}{6n} \times \frac{3n}{8n} \times \frac{3n}{10n} z^5 + \frac{n}{4n} \times \frac{n}{6n} \times \frac{3n}{8n} \times \frac{3n}{10n} \times \frac{5n}{12n} \times \frac{5n}{14n} z^7 +$ &c. Or (becauſe the n every where divides it ſelf) the Length of the Arch $AVM = na = e = z + \frac{1}{4} \times \frac{1}{6} z^3 + \frac{1}{4} \times \frac{1}{6} \times \frac{3}{8} \times \frac{3}{10} z^5 + \frac{1}{4} \times \frac{1}{6} \times \frac{3}{8} \times \frac{3}{10} \times \frac{5}{12} \times \frac{5}{14} z^7$ &c. $= z + \frac{1}{24} z^3 + \frac{3}{640} \times z^5 + \frac{5}{7168} z^7 + \frac{35}{294912} z^9 + \frac{63}{2883584} z^{11} + \frac{231}{54525952} z^{13} + \frac{143}{167772160} z^{15} + \frac{6435}{36507222016} z^{17} +$ &c. *Confectary* 1.

So proceeding the ſame Way, from the 15th Confectary of the laſt Section, for finding the Length of the Arch $VM = ny = \frac{1}{2} e = u$, we have $PM = s = u - \frac{1}{2} \times \frac{1}{3} u^3 + \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5} u^5 - \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5} \times \frac{1}{6} \times \frac{1}{7} u^7 +$ &c. and $VM = ny = u = s + \frac{1}{6} s^3 + \frac{3}{40} s^5 + \frac{5}{112} s^7 + \frac{35}{1152} s^9 + \frac{63}{2816} s^{11} + \frac{231}{13312} s^{13} + \frac{143}{10240} s^{15} + \frac{6435}{557056} s^{17} +$ &c. *Confectary* 2. *

So again by a like Procedure, from the ninth Theorem in the laſt Section, for finding the Length of the Arch $AVM = na = e$ there, we have $x = e - \frac{1}{4d} \times \frac{1}{6d} e^3 + \frac{1}{4d} \times \frac{1}{6d} \times \frac{1}{8d} \times \frac{1}{10d} e^5 - \frac{1}{4d} \times \frac{1}{6d} \times \frac{1}{8d} \times \frac{1}{10d} \times \frac{1}{12d} \times \frac{1}{14d} e^7 +$ &c. And $AVM = na = e = x + \frac{1}{4d} \times \frac{1}{6d} x^3 + \frac{1}{4d} \times \frac{1}{6d} \times \frac{3}{8d} \times \frac{3}{10d} x^5 + \frac{1}{4d} \times \frac{1}{6d} \times \frac{3}{8d} \times \frac{3}{10d} \times \frac{5}{12d}$

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* *Again, for the ſame Reaſons, as before, the Learner may, if he thinks fit, proceed directly from hence to the ſeventh Section.*

$\frac{5}{12d} \times \frac{5}{14d} x^7 + \frac{1}{4d} \times \frac{1}{6d} \times \frac{3}{8d} \times \frac{3}{10d} \times \frac{5}{12d} \times \frac{5}{14d} \times \frac{7}{16d} \times \frac{7}{18d} x^9 +$ &c.

Theorem X.

Hence from the 17th Conſectary of the laſt Section, where $d = 2^1$, and $x = 2 \times \overline{2VP}^{\frac{1}{2}} = 2 \times \overline{2v}^{\frac{1}{2}}$, we have in the *Reſolutum* there, the Length of the Arch A V M $= n a = e = x + \frac{1}{96} x^3 + \frac{3}{10240} x^5 + \frac{5}{458752} x^7 + \frac{35}{75497472} x^9 + \frac{63}{29527900016} x^{11} +$ &c. *Conſectary* 3.

From the 18th Conſectary of the ſame Section, where $d = 2^2$, and $x = 4 \times \overline{2 - \overline{4 - 2v}^{\frac{1}{2}}}^{\frac{1}{2}}$, we have in the *Reſolutum*, the Length of the Arch A V M $= n a = e = x + \frac{1}{384} x^3 + \frac{3}{163840} x^5 + \frac{5}{29360128} x^7 + \frac{35}{19327352832} x^9 + \frac{63}{3023656976384} x^{11} +$ &c. *Conſectary* 4.

And thus we may proceed on to ſtill greater Exactneſs, as in the 19th and 20th Conſectaries of the laſt Section, *&c.* Whence it is plain, that a *Reſolutum* may be formed for finding the Length of any Arch A V M of a Circle, to whatever aſſigned Exactneſs we pleaſe, and by as few of the firſt Terms of the Series as we pleaſe. A Diſcovery, I conceive, vaſtly preferable to to any Thing ever yet invented for the ſame Purpoſe. The Excellence of which will appear from an Example to the laſt Conſectary above; where d riſes no higher thau $2^2 = 4$.

Suppoſe then the Length of the Arch A V M ſought to be no ſmaller than than that of 60 Degrees, or one ſixth Part of the Circumference of the Circle; whoſe Radius is V R $= c = 1$; and conſequently the given Chord A M $= z = 1$. Whence V P $= v = 1 - \overline{1 - \frac{1}{4}}^{\frac{1}{2}}. = 0.133974596 2156$, $x = 4 \times \overline{2 - \overline{4 - 2v}^{\frac{1}{2}}}^{\frac{1}{2}} = 1.04$

420953776O4; $x^3 = 1.13857846977$, $x^5 = 1.241475858$, $x^7 = 1.3536725$; $x^9 = 1.47601$, $x^{11} = 1.609$. Then putting $A = \frac{1}{384}$, $B = \frac{3}{163840}$, $C = \frac{5}{29360128}$, $D = \frac{35}{19327352832}$, $E = \frac{63}{3023656976384}$, &c. we have

$$\begin{array}{lr}
+\ z & = 1.04420953776O4 \\
+\ Az^3 & 29650480983 \\
+\ Bz^5 & 227321019 \\
+\ Cz^7 & 2305290 \\
+\ Dz^9 & 26729 \\
+\ Ez^{11} & 335 \\
\hline
\end{array}$$

Sum $= 1.047197551196O =$ the Length of the Arch A V M of the ſixth Part of a Circle, whoſe Radius is Unity, true to the twelfth Dicimal incluſively; without making Uſe of any more than the ſix firſt Terms of the Series only.

And by ſtill proceeding on from the tenth Theorem here, according to the Directions in the fourth and fifth Sections, we may have the Value of the Arch A V M to any aſſigned Exactneſs, by only a few continued Extractions of the Square Root; without making Uſe of any more Terms in the Series than the firſt two $x + \frac{xxx}{24dd}$, or even than the firſt one x.

In the third Conſectary, let $x = \overline{2v}^{\frac{1}{2}}$; and we ſhall have the Length of the Arch A V M $= e = 2$ multiqlying the Series $x +$

$\frac{1}{24}x^3 + \frac{3}{640}x^5 + \frac{5}{7168}x^7 + \frac{35}{294912}x^9 + \frac{63}{2883584}x^{11} +$ &c. (as in the firſt Confectary) for *Confectary* 5.

In the fourth Confectary, let $x = \overline{2 - \overline{4 - 2v}^{\frac{1}{2}}}^{\frac{1}{2}}$; and we ſhall have AVM=e=4 multiplying the Series $x + \frac{1}{24}x^3 + \frac{3}{640}x^5 + \frac{5}{7168}x^7 +$ &c. (again as in the firſt Confectary) for *Confectary* 6.

SECT. VII.

Of the Reduction of Equations to an Ultimum, *and of the Doctrine of* Ultimators.

I Now proceed to one Property belonging to the Expreſſion of of a compound Equation, which ſeems never to have been ſo much as thought of before: tho' it be the only true Baſis of that late admirable Invention, which is very improperly called the Doctrine of *Fluxions* by ſome, and the *Differential Calculus* by others.

If, for Example, it were required to find the Root of the fifth Power from the Equation $154a - 11a^2 - 33a^3 + 11a^4 - a^5 = 120$, or $120 - 154a + 11a^2 + 33a^3 - 11a^4 + a^5 = 0$, the required Root a would be found to be Unity, or $1 - a = 0$: Which dividing $120 - 154a + 11a^2 + 33a^3 - 11a^4 + a^5 = 0$ quotes $120 - 34a - 23a^2 + 10a^3 - a^4 = 0$. Then the Biquadratic Root from the laſt Biquadratic Equation may be found to be $a = 3$, or $3 - a = 0$: Which dividing $120 - 34a - 23a^2 + 10a^3 - a^4 = 0$ quotes $40 + 2a - 7a^2 + a^3 = 0$. Again, the Cubic Root from that may be found to be $a = 5$, or $5 - a = 0$: Which dividing $40 + 2a - 7a^2 + a^3 = 0$ quotes $8 + 2a - a^2 = 0$. Laſtly, the Square Root from that may be found to be $a=4$, or $4-a=0$: Which dividing $8+2a-a^2=0$ quotes $2+a=0$, whence $a=-2$. Therefore, which ſoever of theſe Roots firſt comes out, the Equation of the fifth Power $120-154a+11a^2+33a^3-11a^4+a^5=0$ will be found to be a Product made by the continual Multiplication of $2+a=0$ by $4-a=0$ by $5-a=0$ by $3-a=0$ by $1-a=0$: And conſequently its five Roots are -2, $+4$, $+5$, $+3$, $+1$. The Biquadratic Equation $120-34a-23a^2+10a^3-a^4=0$ will be found to be a Product made by the Multiplication of $2+a=0$ by $4-a=0$ by $5-a=0$ by $3-a=0$: Conſequently its four Roots are -2, $+4$, $+5$, $+3$. The Cubic Equation $40+2a-7a^2+a^3=0$ will be found a Product made by the Multiplication of $2+a=0$ by $4-a=0$ by $5-a=0$: Conſequently, its three Roots are -2, $+4$, $+5$. The Quadratic Equation $8+2a-a^2=0$ will be found to be a Product made from the Multiplication of $2+a=0$ by $4-a=0$: Whence its two Roots are -2, $+4$. And thus it will always appear, that every Equation conſiſts of as many Roots as it has Dimenſions. Theorem XI.

And every ſuch explicable Equation muſt alſo have as many affirmative Roots, as it has Changes of its Signs from + to —, and from — to +; the reſt being Negative. Thus in the Equation of the fifth Power above, the Signs in their Order, are + — + + — +; which ſhews that there are four affirmative Roots. In the Biquadratic Equation above, the Signs are + — — + —; which ſhews that there are three affirmitive Roots. And ſo for any other. Theorem XII.

If in the foregoing Equations we put $n=2$, $p=4$, $q=5$, $r=3$, $s=1$. Then the Quadratic Equation, or Compound of $\overline{n+a=0} \times \overline{p-a}=0$, will be $npa \begin{matrix}+p\\-n\end{matrix} a-a^2=0$; which again multiplied by $q-a=0$ gives the Cubic Equation $npqa^0 \begin{matrix}+pq\\-nq\\-np\end{matrix}$ $a \begin{matrix}-q\\-p\\+n\end{matrix} a^2+a^3=0$, which again multiplied by $r-a=0$ gives the Biquadratic Equation $$npqra^0 \begin{matrix} +pqr \\ -nqr \\ -npr \\ -npq \end{matrix} a^1 \begin{matrix} -qr \\ -pr \\ +nr \\ -pq \\ +nq \\ +np \end{matrix} a^2 \begin{matrix} +q \\ +p \\ -n \\ +r \end{matrix} a^3 - a^4 = 0.$$

Where the Roots are $-n$, $+p$, $+q$, $+r$. And thus, if in any Inſtance we proceed ever ſo high, we ſhall always find, that the Coefficient of the laſt Term equals Unity, the Coefficient of the laſt Term but one is equal to the Sum of all the Roots under their proper Signs; the Coefficient of the laſt Term but two is equal to the Sum of the Products of every two of the Roots, taking each Product under its contrary Sign; the Coefficient of the laſt Term but three, is equel to the Sum of the Products of every three of the Roots, taking each Product under its proper Sign; the Coefficient of the laſt Term but four is equal to the Sum of the Products of every four of the Roots, taking each Product under its contrary Sign; the Coefficient of the laſt Term but five is equal to the Sum of the Products of every five of the Roots, taking each Product under its proper Sign, *&c.* THEOREM XIII.

Alſo according to this Method of Procedure, in all Equations, the firſt Term will be affirmatixe; the laſt Term will be of the ſame Sign with that of the Product of all the Roots, taking each Root under its contrary Sign; and every intermediate Term will be of the ſame Sign with its Coefficient found as above. THEOREM XIV.

Therefore every Algebraic Equation univerſally, conſiſting of only one unknown Quantity a being made equal to nothing (and A put for the abſolute Number or Coefficient of $a^0=1$, B for the Coefficient of a^1 or a, C for the Coefficient of a^2, D for the Coefficient of a^3, E for the Coafficient of a^4, *&c.* alphabetically) will be repreſented by $Aa^0 \pm Ba^1 \pm Ca^2 \pm Da^3 \pm Ea^4 \pm Fa^5 \pm$ *&c.* $=0$. And whenever any Power of a is wanting, we are to ſuppoſe its proper Coefficient equal to nothing. Thus, if a^1 be wanting, $B=0$; if a^2 be wanting, $C=0$; if a^3 be wanting, $D=0$, *&c.* Alſo upon the ſame Suppoſition, if we divide the whole

Equation by the Coefficient of the higheſt Power of a; then will the Coefficients be the ſame with thoſe of the foregoing Examples under the following univerſal Forms, *viz.* every firſt Power will be expreſſed by $\frac{A}{B}a^0 \pm a^1 = 0$, every Quadratic Equation by $\frac{A}{C}a^0 \pm \frac{B}{C}a^1 \pm a^2 = 0$, every Cubic by $\frac{A}{D}a^0 \pm \frac{B}{D}a^1 \pm \frac{C}{D}a^2 \pm a^3 = 0$, every Biquadratic by $\frac{A}{E}a^0 \pm \frac{B}{E}a^1 \pm \frac{C}{E}a^2 \pm \frac{D}{E}a^3 \pm a^4 = 0$, *&c.* and every n Power, the Coefficient of whoſe laſt Terms is Z, by $\frac{A}{Z}a^0 \pm \frac{B}{Z}a^1 \pm \frac{C}{Z}a^2 \pm \frac{D}{Z}a^3 \pm \frac{E}{Z}a^4 \pm$ *&c.* $\pm a^n = 0$.

If the Roots be every one of them equal to $+c$ or $-c$; then every Quadratic Equation will be equal to $\overline{c \pm a}^2 = 0$, every Cubic to $\overline{c \pm a}^3 = 0$, every Biquadratic to $\overline{c \pm a}^4 = 0$, *&c.* as in the Scale of the third Section. And every n Power will be equal to $\overline{c \pm a}^n = c^n \pm nc^{n-1}a + n\times\frac{n-1}{2}c^{n-2}a^2 \pm n\times\frac{n-2}{2}\times\frac{n-2}{3}c^{n-3}a^3 +$ &c. $\pm a^n = 0$, THEOREM XV.

Hence it appears, that in every Equation, if the abſolute known Quantity $A = Aa^0$ be required to be brought to the greateſt, or (conſequently, when the ſame is conſidered as the Complement or Remainder to another Quantity) the leaſt Value it poſſibly can have, while all the remaining Parts of the Expreſſion continue the ſame; we muſt bring all the Roots or Values of a, according to its higheſt Dimenſion, to an Equality with one another. For, if, for Inſtance, the two laſt Terms of the Expreſſion containing the higheſt Dimenſions of a be $\pm Ya^{n-1} \pm Za^n$; then after the Whole is divided by Z we have (Theorem 13.) $\frac{Y}{Z}$ equal to the Sum of the Roots under their proper Signs.

And the Number of all the Roots is n (Theorem 11.): Whence the Root a itſelf muſt be equal to $\frac{Y}{nZ}$ But the Coefficient $\frac{Y}{Z}$ is here a fixed given Number, and if its n Number of Parts be unequal to each other, it is plain, they can never make ſo great a Product by the Multiplication of all together, nor conſequently that Product ſubtracted from the ſame given Quantity ſo little a Remainder, as when the ſame $\frac{Y}{Z}$ is divided into the ſame Number n of Parts all equal to each other. It follows therefore, that in every Equation, if it be required to bring the abſolute Quantity A to its greateſt or leaſt poſſible Value, we muſt bring all the Roots of the Equation to the ſame Value. Theorem XVI.

This is what I call *the Reduction of an Equation to its Ultimate.* And in this Reſpect, I call every ſeparate Term in the Equation a *Subject*: To which each Term in the Equation, after the Reduction is performed, may be called its reſpective *Ultimator*. Alſo to diſtinguiſh variable or unknown Quantities from ſuch as are always invariably the ſame, let the former be repreſented by ſome one or more of the Letters, a, e, s, t, *v*, u, x, y, z; and the latter by b, c, d, f, g, h, k, l, m, n, p, q, r. Otherwiſe, the latter, may be alſo repreſented by the Capitals A, B, C, D, E, *&c.*

Example I. Let it be required to reduce the Quadratic Equation $Aa^0 \pm Ba^1 \pm Ca^2 = 0$ to its Ultimate.

Here $A = Ba \pm Ca^2$, or $\frac{A}{C} = \frac{B}{C} a \pm a^2$. Whence (Theo. 16.) to bring $\frac{A}{C}$ to its ultimate Value, we muſt make $\frac{B}{C} = \mp 2c$ in the Square of $c \pm a = 0$; or, which is the ſame, make $\frac{B}{2C} \pm a$

$= 0$.

$= 0$. Therefore the the Ultimate required is $B \pm 2Ca = 0$, or $Ba^{\circ} \pm 2Caa^{\circ} = 0$.

Example II. Let it be required to reduce any Equation to its Ultimate, which conſiſts of but one Power of the Variant beſides the firſt. For Example, the Equation $A \pm Ba \pm Za^{n} = 0$.

Here $A = Ba \pm Za^{n}$, or $\frac{A}{Z} = \frac{B}{Z} a \pm a^{n}$. Whence (Theorem 16) to bring $\frac{A}{Z}$ to its ultimate Value, we muſt make $\frac{B}{Z} = \mp n c^{n-1}$ in the n Power of $c \pm a = 0$. That is (becauſe $c = a$) $\frac{B}{Z} = n a^{n-1}$. Therefore the Ultimate required is $B \pm nZa^{n-1} = 0$, or $Ba^{\circ} \pm nZa^{n-1}a^{\circ} = 0$.

Example II. Let it be required to reduce the Equation $A \pm Pa^{m} \pm Za^{n} = 0$ to its Ultimate.

Put $a^{m} = e$, whence $a = e^{\frac{1}{m}}$, and $a^{n} = e^{\frac{n}{m}}$. Then again put $\frac{n}{m} = r$. Whence for $Pa^{m} \pm Za^{n}$ we have $Pe \pm Ze^{r}$, whoſe Ultimate by the laſt is $Pe^{\circ} \pm rZe^{r-1}e^{\circ} = 0$, or $Pe \pm rZe^{r} = 0$. Therefore, reſtoring the Values of e and r, we have $Pa^{m} \pm \frac{n}{m} Za^{n} = 0$, or (multiplying by m) $mPa^{m} \pm nZa^{n} = 0$. Conſequently (to keep always the ſame Form) the Ultimate requir'd is $mPa^{m-1}a^{\circ} \pm nZa^{n-1}a^{\circ} = 0$.

Hence it appears, that the Ultimate of the Sum or Difference of any two Equations is the ſame with the Sum or Difference of their reſpective Ultimates. For Inſtance, the Equations $b - Da^{m} + Ea^{n} = 0$ and $c - Ba^{1} + Da^{m} = 0$ have their Sum equal to $\overline{b+c} - Ba^{1} + Ea^{n} = 0$. So their reſpective Ultimates $-m$

$D a^{m-1} a^{\circ} + n E a^{n-1} a^{\circ} = 0$ and $- l B a^{l-1} a^{\circ} + n E a^{n-1} a^{\circ} = 0$ have their Sum equal to $- l B a^{l-1} a^{\circ} + n E a^{n-1} a^{\circ} = 0$: The Ultimators of the abſolute known Quantity A, b, or $b + c$, being always the ſame, *i.e.* equal to nothing. Conſequently whatever invariable Quantity in an Equation is added to or ſubtracted from a variable one, the Ultimator is ſtill the ſame. Thus $2 a a^{\circ}$ is equally the Ultimator of a^2, of $a^2 + b^2$, and of $a^2 - b^2$. So $-2 a a^{\circ}$ is the Ultimator of both $-a a$, and $b b - a a$. Whence it neceſſary follows, that though two Ultimators may be equal, yet we cannot infer from thence that their Subjects are equal. THEOREM XVII.

And if, as has been proved, the Ultimate of the Sum of any two Equations be thus the ſame with the Sum of their reſpective Ultimates, it neceſſarily follows, that the Ultimate of the Sum of never ſo many Equations is the ſame with the Sum of their reſpective Ultimates. Conſequently, if an Equation made up of the Sum of never ſo many other Equations be univerſally repreſented by $A \pm B a \pm C a^2 \pm D a^3 \pm E a^4 \pm F a^5 \pm$ &c. $= 0$ its Ultimate will be $0 \pm B a^{\circ} \pm 2 C a a^{\circ} \pm 3 D a^2 a^{\circ} \pm 4 E a^3 a^{\circ} \pm 5 F a^4 a^{\circ} \pm$ &c, $= 0$. Whence ariſe the following Theorems.

In every Equation, the Ultimator of the Sum or Difference of any two Subjects is the ſame reſpectively with the Sum or Difference of their reſpective Ultimators. THEOREM XVIII.

In every Equation conſiſting only of different poſitive Powers of the ſame Variant, the Ultimator of each Term or Subject is had by multiplying it by its reſpective Exponent, and afterwards ſubtracting Unity from its Exponent, and ſupplying its Place with Unity, as a Power of ſuch Variant, keeping the ſame Sign. THEOREM XIX. And convertibly

The Subject of every Ultimator conſiſting of the ſame Variant, (whoſe Power is poſſitive) is had by taking away its 0 Power, and afterwards adding Unity to its Exponent, and dividing the Whole by the Exponent ſo encreaſed, keeping the ſame Sign. THEOREM XX.

Thus, the Equation $A\,a^{\circ} \pm B\,a^{\frac{1}{2}} \pm C\,a^{\frac{1}{3}} \pm D\,a^{\frac{1}{4}} \pm E\,a^{\frac{1}{5}} \pm$ &c. $= 0$ has the Ultimate $0 \pm \frac{1}{2} B\,a^{\frac{1-2}{2}}\,a^{\circ} \pm \frac{1}{3} C\,a^{\frac{1-3}{3}}\,a^{\circ} \pm \frac{1}{4} D\,a^{\frac{1-4}{4}}\,a^{\circ} \pm \frac{1}{5} E\,a^{\frac{1-5}{5}}\,a^{\circ} \pm$ &c. $= 0$, or $\pm \frac{Ba^{\circ}}{2a^{\frac{1}{2}}} \pm \frac{Ca^{\circ}}{3a^{\frac{2}{3}}} \pm \frac{Da^{\circ}}{4a^{\frac{3}{4}}} \pm \frac{Ea^{\circ}}{5a^{\frac{4}{5}}} \pm$ &c. $= 0$. Alſo the Equation $A\,a^{\circ} - B\,a^{\frac{c}{m}} \pm C\,a^{\frac{r}{n}} = 0$, has the Ultimate $0 - \frac{c}{m} B\,a^{\frac{c-m}{m}}\,a^{\circ} + \frac{r}{n} C\,a^{\frac{r-n}{n}}\,a^{\circ} = 0$, or $-\frac{c\,B\,a^{\frac{c}{m}}a^{\circ}}{m\,a} + \frac{r\,C\,a^{\frac{r}{n}}\,a^{\circ}}{n\,a} = 0$.

When a Subject conſiſts of ſeveral Terms with the ſame Exponent in common, we muſt raiſe it to the Power always of the Numerator of the Exponent; by which Means the Numerator of the Exponent will always be Unity, and afterwards proceed as follows. For Example, let the Subject be $\overline{b\,x^{m} \pm c\,y^{n}}^{\frac{1}{r}}$. Put $\overline{b\,x^{m} \pm c\,y^{n}}^{\frac{1}{r}} = z$. Whence $b\,x^{m} \pm c\,y^{n} = z^{r}$. Conſequently (Theo. 19.) $m\,bx^{m-1}\,x^{\circ} \pm n\,c\,y^{n-1}\,y^{\circ} = r\,z^{r-1}\,z^{\circ}$; whence $\frac{mbx^{m-1}x^{\circ} \pm ncy^{n-1}v^{\circ}}{rz^{r-1}} = z^{\circ}$. But $z^{r-1} = \frac{z^{r}}{z} = \frac{\overline{bx^{m} \pm cy^{n}}^{1}}{\overline{bx^{m} \pm cy^{n}}^{\frac{1}{r}}} = \overline{bx^{m} \pm cy^{n}}^{\frac{r-1}{r}}$. Therefore $\frac{mbx^{m-1}\,x^{\circ} \pm ncy^{n-1}y^{\circ}}{r \times \overline{bx^{m} \pm cy^{n}}^{\frac{r-1}{r}}} = z^{\circ}$, which being the Ultimator of z muſt therefore be the Ultimator to the given Subject. Hence

Whenever the Numerator of the Exponent of a Subject is Unity and poſitive. Let the given Subject divide its firſt Power, and multiply the Quotient by the Denominator of the given Exponent. Then with that Product divide the Ultimator of the firſt Power of the Subject, and the Quotient will be the Ultimator required. THEOREM XXI.

Proceed we next to ſhew how the Ultimators muſt be found to Products made up of different Variants. In the firſt Place then, let it be required to find the Ultimator of the Product ae conſiſting of only two Variants. Put $ae = bee$, and $ae = caa$. Whence $a = be$, and $e = ca$, and $ae = \frac{1}{2}bee + \frac{1}{2}caa$. The Ultimator of which laſt is (Theo. 19.) $bee^\circ + caa^\circ$, and conſequently is equal to the Ultimator of ae. But $be = a$, and $ca = e$. Therefore theſe ſubſtituted for their Equals in that Ultimator give $ae^\circ + ea^\circ$ for the Ultimator of ae.

Hence it will be eaſy to find the Ultimator of any Fraction $\frac{x}{z} = y$, whence $x = yz$: and by the laſt Example $x^\circ = yz^\circ + zy^\circ$, or $x^\circ - yz^\circ = zy^\circ$, or $\frac{x^\circ - yz^\circ}{z} = y^\circ$. But $y = \frac{x}{z}$, therefore $\frac{x^\circ - \frac{x}{z}z^\circ}{z} = \frac{zx^\circ - xz^\circ}{zz} = y^\circ$ the Ultimator of $\frac{x}{y}$.

And hence again, to find the Ultimator of a Product $aez = x$ of three Letters. Here $ae = \frac{x}{z}$, and $ae^\circ + ea^\circ = \frac{zx^\circ - xz^\circ}{zz}$, or $z^2ae^\circ + z^2ea^\circ = zx^\circ - xz^\circ$, or $z^2ae^\circ + z^2ea^\circ + xz^\circ = zx^\circ$, or again (ſubſtituting aez for x) $z^2ae^\circ + z^2ea^\circ + aezz^\circ = zx^\circ$. Then dividing by z, we have $zae^\circ + zea^\circ + aez^\circ = x^\circ$ the Ultimator of aez.

So to find the Ultimator of $aesz = x$. Here $aes = \frac{x}{z}$, and $aes^\circ + ase^\circ + esa^\circ = \frac{zx^\circ - xz^\circ}{zz}$, or $z^2aes^\circ + z^2ase^\circ + z^2esa^\circ = zx^\circ - xz^\circ$, or $x^2aes^\circ + z^2ase^\circ + z^2esa^\circ + xz^\circ = zx^\circ$, or (again ſubſtituting $aesz$ for x) $z^2aes^\circ + z^2ase^\circ + z^2esa^\circ + aeszz^\circ = zx^\circ$.‸ dividing by z, we have $zaes^\circ$ ‸ Then $+ zaese^\circ + zesa^\circ + aesz^\circ = x^\circ =$ the Ultimator required.

And thus, it is eaſy to conceive, it will always be, proceed we never ſo high. Alſo it is eaſy to conceive, that as by making $aes = y^3$, whence $aes^{\circ} + ase^{\circ} + esa^{\circ} = 3y^2y^{\circ}$ and $zaes^{\circ} + zase^{\circ} + zesa^{\circ} = 3zy^2y^{\circ}$, and by ſubſtituting Equals, the laſt Ultimator is thus reduced to $3zy^2y^{\circ} + y^3z^{\circ} = x^{\circ}$; ſo any other Subject $bae^m s^n$ will have the Ultimator $nbae^m s^{n-1}s^{\circ} + mbas^n e^{m-1}e^{\circ} + be^m s^n a^{\circ}$. Whence

As often as the ſame Variant Occurs in a Product, ſo often in the Ultimator muſt the Product of all the reſt occur with the Ultimator of that Variant. THEOREM XXII.

Let the Fraction $\frac{x^m}{z^n} = y$. Whence $x^m = z^n y$, and by the laſt $mx^{m-1}x^{\circ} = nyz^{n-1}z^{\circ} + z^n y^{\circ}$, or $mx^{m-1}x^{\circ} - nyz^{n-1}z^{\circ} = z^n y^{\circ}$, (or ſubſtituting $\frac{x^m}{z^n}$ for y) $mx^{m-1}x^{\circ} - n\frac{x^m}{z^n}z^{n-1}z^{\circ} = z^n y^{\circ}$, or $mx^{m-1}x^{\circ} - n\frac{x^m}{z}z^{\circ} = z^n y^{\circ}$. Therefore (dividing by z^n) we have $\frac{mzx^{m-1}x^{\circ} - nx^m z^{\circ}}{z^{n+1}} = y^{\circ} =$ the Ultimator of $\frac{x^m}{z^n}$. Which gives the following Theorem for finding the Ultimators of all Fractions.

Firſt, Multiplying the Ultimator of the Numerator, by the firſt Power of the Denominator, and call that the firſt Product. *Secondly*, Multiply together the Numerator, the Exponent of the Denominator, and the Ultimator of the firſt Power of the Denominator, and call that the ſecond Product. *Thirdly*, Multiply the Denominator by its firſt Power, and call that the third Product. Then ſubtract the ſecond Product from the firſt, and the Remainder will be the Numerator, and the third Product the Denominator of the Ultimator required. THEOREM XXIII.

For Example, let a Fraction be $\frac{p - xx}{\overline{d-x}^{\frac{2}{3}}}$. Here the firſt Product is $-2xx^{\circ} \times \overline{d-x} = -2dxx^{\circ} + 2x^2x^{\circ}$. The ſecond

Product is $p - x\,x \times \frac{2}{3} x - x^{\circ} = -\frac{2}{3} p x^{\circ} + \frac{2}{3} x^{2} x^{\circ}$. The third Product is $\overline{d-x}^{\frac{2}{3}} \times \overline{d-x} = \overline{d-x}^{\frac{5}{3}}$. And the ſecond Product ſubtracted from the firſt leaves $-2dxx^{\circ} + \frac{2}{3} p x^{\circ} + \frac{4}{3} x^{2} x^{\circ}$. Therefore the Ultimator required is $\frac{\frac{2}{3}px^{\circ} - 2dxx^{\circ} + \frac{4}{3}x^{2}x^{\circ}}{\overline{d-x}^{\frac{5}{3}}}$.

In negative Powers put $ba^{-n} = \frac{b}{a^{n}} = z$. Whence $b = a^{n} z$, and (Theo. 17, 19.) $o = nza^{n-1}a^{\circ} + a^{n}z^{\circ}$, or $-nza^{n-1}a^{\circ} = a^{n}z^{\circ}$, or (ſubſtituting $\frac{b}{a^{n}}$ for z) $-n\frac{b}{a}a^{\circ} = a^{n}z^{\circ}$, (or dividing by a^{n}) $-\frac{nba^{\circ}}{a^{n+1}} = -nba^{-n-1}a^{\circ} = z^{\circ}$ the Ultimator of ba^{-n}.

Whence it appears, that in Negative Powers the Subject and its Ultimator are always of contrary Signs, but in every other Reſpect are directly formed from each other, as in the 19th and 20th Theorems. THEOREM XXIV.

Next for the Ultimators of Logarithims. It is known that the natural Logarithm of any Number repreſented by $1 + z$ is $z - \frac{1}{2}z^{2} + \frac{1}{3}z^{3} - \frac{1}{4}z^{4} + \frac{1}{5}z^{5} - \frac{1}{6}z^{6} +$ *&c.* the ſame with na in the *Reſolutum* to Conſect. 1. Sect. IV. Putting the c there equal to Unity and the n infinite, and proceeding afterwards as in Conſect. 1. Sect. VI. But the Ultimator of that Logarith is $z^{\circ} - zz^{\circ} + z^{2}z^{\circ} - z^{3}z^{\circ} + z^{4}z^{\circ} - z^{5}z^{\circ} +$ &c. $= \frac{z^{\circ}}{1+z}$, as appears from Article 6. Sect. I. (putting 1 for the a there, z for e, and z° for c). Therefore the Ultimator of the Logarithm of every Number $1+z$ is $\frac{z^{\circ}}{1+z}$. THEOREM XXV.

Hence it is plain from the Nature of Logrithms. that the Ultimator of the Logarithm of the n Power of any Number $1+z$ is $\frac{nz^{\circ}}{1+z}$. That is, the Ultimators of the Logarithms of all Powers of the ſame Number are to another, as the Exponents of thoſe Powers; or as the Logarithms themſelves. THEOREM XXVI.

Hence again we are taught the Method of determining the Ultimators of Subjects, whoſe Exponents themſelves are variable; which are therefore called *Exponential Quantities*, and are of different Degrees, according as the Exponents themſelves are involved. Thus, the Quantity, whoſe variable Exponent is without an Exponent itſelf, as a^{e}, is called an *Exponential of the firſt Degree.* But when the Exponent itſelf is an Expotential of the firſt Degree, as $a^{e^{s}}$, the Quantity is called an *Evponential of the ſecond Degree.* And when that Exponent agrin is an Exponential of the firſt Degree, as $a^{e^{s^{v}}}$, the Quantity is called an *Exponential of the third Degree.* And ſo on.

But obſerve, that ſince in theſe Operations we are obliged to make Uſe of Logarithms, let the Logarithm of every Quantity be denoted by the Capital L ſet before it, and its Ultimator by $\overset{\circ}{\mathrm{L}}$. Thus $\mathrm{L.\,c^{x}}$ denotes the Lagarith of of $\mathrm{c^{x}}$, and $\overset{\circ}{\overline{\mathrm{L.\,c^{x}}}}$ its Ultimator. And again obſerve, that ſince z° is the Ultima^tor of $1+\mathrm{z}$ (Theo. 17.) and the Ultimator of the Logarithm of every Number $1+\mathrm{z}$ is $\frac{\mathrm{z}^{\circ}}{1+\mathrm{z}}$ (Theo. 25); therefore the Ultimator of the Logarithm of every other variable Quantity a is $\frac{\mathrm{a}^{\circ}}{\mathrm{a}}$. THEOREM XXVII.

Example 1. Let it be required to find the Ultimator of $\mathrm{c^{x}}$, where c is invariable. Put $\mathrm{c^{x}} = \mathrm{a}$. Whence $\mathrm{L.\,c^{x}} = \mathrm{L.\,a}$. But (Theo. 26.) $\mathrm{x} : 1 = \mathrm{L.\,c^{x}} : \mathrm{L.\,c}$, whence $\mathrm{L.\,c^{x}} = \mathrm{x} \times \mathrm{L.\,c} = \mathrm{L.\,a}$; and conſequently (Theo. 22.) $\mathrm{x}^{\circ} \times \mathrm{L.\,c} + \overset{\circ}{\overline{\mathrm{L.\,c}}} = \overset{\circ}{\overline{\mathrm{L.\,a}}}$, or becauſe (Theo. 17.) $\overset{\circ}{\overline{\mathrm{L.\,c}}} = 0$; therefore $\mathrm{x}^{\circ} \times \mathrm{L.\,c} = \overset{\circ}{\overline{\mathrm{L.\,a}}} = \frac{\mathrm{a}^{\circ}}{\mathrm{a}}$ (Theo. 27). But $\mathrm{c^{x}} = \mathrm{a}$, therefore $\mathrm{x}^{\circ} \times \mathrm{L.\,c} = \frac{\mathrm{a}^{\circ}}{\mathrm{c^{x}}}$. Conſequently, $\mathrm{c^{x}} \times \mathrm{L.\,c} \times \mathrm{x}^{\circ} = \mathrm{a}^{\circ}$ is the Ultimator of $\mathrm{c^{x}}$ required.

Example II. Let it be required to find the Ultimator of z^y, where z is variable. Put $z^y = a$, whence (Theo. 26.) $L.\,z^y = L\,a = y \times L.\,z$, and (Theo. 27.) $\overline{L.\,a}^{\circ} = \frac{a^{\circ}}{a} = y^{\circ} \times L.\,z + y \times \overline{L.\,z}^{\circ} = y^{\circ} \times L.\,z + \frac{y}{z}\,z^{\circ}$. Conſequently, ſince $a = z^y$, we have $a^{\circ} = z^y \times \overline{L.\,z \times} y^{\circ} + y\,z^{y-1}\,z^{\circ} =$ the Ultimator of z^y required.

Example III. Let it be required to find the Ultimator of $z^{y^x} = a$. Here $L.\,z^{y^x} = L.\,a$, and (Theo. 26.) $y^x \times L.\,z = L.\,a$. But the Ultimator of y^x, by the laſt Example, is $y^x \times \overline{L.\,y \times}\,x^{\circ} + x\,y^{x-1}\,y^{\circ}$: therefore the Ultimator of $y^x \times L.\,z$ is $y^x \times L.\,z \times \overline{L.\,y} \times x^{\circ} + L.\,z \times x\,y^{x-1}\,y^{\circ} + y^x \times \frac{z^{\circ}}{z} = \overline{L.\,a}^{\circ} = \frac{a^{\circ}}{a} = \frac{a^{\circ}}{z^{y^x}}$. Whence $z^{y^x} \times y^x \times \overline{L.\,y} \times x^{\circ} + L.\,z \times z^{y^x} \times x\,y^{x-1}\,y^{\circ} + z^{y^x}\,y^x \times \frac{z^{\circ}}{z} = a^{\circ} =$ the Ultimator of z^{y^x} required.

And thus by ſtill making the foregoing Example a Step to the next, it will be eaſy to conceive how we may aſcend to what Height we pleaſe.

Now from all that has been hitherto inferred from Theo. 16. we ſee how neceſſary it is always to preſerve each Variant in the Ultimator under its o Power. For firſt of all it is plain, that without this we cou'd have no Means from the Nature of the Thing itſelf, whereby to diſtinguiſh an Ultimator from a Subject. Whereas in every original Equation brought to the ſame Variant with ſuch Variant in any Term of it under its firſt Power (or when it has not ſuch a firſt Power of its Variant, by bringing it to another Variant, which has; as was done in the third Example to the 16th Theorem above); it appears that there is always this univerſal Difference between every Subject and its Ultimator, that in the former, the Exponent of the Variant is greater by Unity than in the latter. And afterwards by thus multiplying, as we have done, the whole Ultimate by the o Power of its Variant, it appears that we are thereby able (which we otherwiſe cou'd not be) to ſubſtitute in that Ultimate the Ultimator of the Product or Quotient of any two or more other Variants, which may be equal to the Ultimator of any Term of the given Equation; by which Means only we can determine the Ratios, which the Ultimate Values of different Variants bear to one another, according to the Conditions of the given Equation. For it is evident (from the Inveſtigation of Theo. 22 and 23,) that we cannot otherwiſe be able to expreſs the Ultimator of ſuch a Product or Quotient, agreeable to the necessary Laws of the Reduction of an Equation to its Ultimate.

All which is no more than what neceſſarily reſults from this Conſideration of Unity as the 0 Power of every Quantity. For Example, in any Ultimate Equation including the two different Variants a and e; ſuppoſe, after their Values are found, we have $a=2x$ and $e=3x$; then, it is plain, if we put $a=1$, whence $e=\frac{3}{2}$, we have $a^0=1=2x$: and if we put $e=1$, whence $a=\frac{2}{3}$, we have $e^0=1=3x$. Hence it follows, that in that Equation, a^0 is in Power to e^0 as 2 to 3; tho' at the ſame Time it be certain, that in Quantity a^0 is to e^0 only as an Unit is to an Unit.

Since therefore, as has ſufficiently appeared, the whole Operation of reducing an Equation to its Ultimate, depends upon the different Ratio of the Powers of its Variants to one another; and ſince the Expreſſions by theſe 0 Powers do actually as much imply the Difference of ſuch Ratios as thoſe by the ſame Variants under any other Exponent: It neceſſarily follows, that to take away theſe Expreſſions, by ſubſtituting Unity in their Stead, wou'd bring as much Confuſion to the Work, as to ſuppoſe any other different Variants the ſame, when they chance to have the ſame Exponent.

It is evident then, as often as any Subject conſiſts of different Variants *Ex. gr.* x, y, z, that the Expreſſions x^0, y^0, z^0, in its Ultimator have the ſame Difference in Power with the ſame Variants under any other common Exponent x^n, y^n, z^n; the firſt as beginning the Scale x^0, x^1, x^2, x^3, &c. the ſecond as beginning the Scale y^0, y, y^2, y^3, &c. and the third as beginning the Scale z^0, z^1, z^2, z^3, *&c.* When perhaps they have no other common Exponent through the whole given Equation. Therefore the Expreſſion x^0, y^0, z^0, I conceive may be each fitly called the *Peculiar Unit* of its reſpective Scale of Powers.

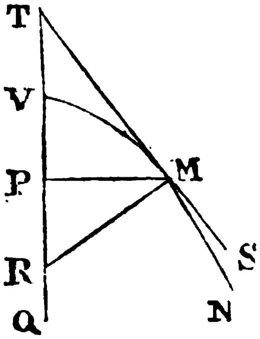
Hence every *Ultimator* may be defined to be, *The proper Reference of each Subject in a given Equation to the* Peculiar Units *of the Powers of all its Variants, in Order to diſcover the Ratios of thoſe Variants to one another in their Ultimate State.* Which I take to be the true Definition of what has been hitherto moſt improperly and unintelligibly called a *Fluxion* by ſome, and a *Differential* by others.

And here take Notice, that, in Conformity with the Method of *Fluxions,* I call the finding of an Ultimator from its given Subject, *The direct Method of Ultimators*; and contrarywiſe, the finding of a Subject from its given Ultimator, I call, *The Inverſe Method of Ultimators.* Examples in each of which I ſhall give promiſcuouſly in the two following Sections; but, according to my deſigned Brevity, chiefly in ſuch Caſes, where I think Things might be delivered in a much perfecter Manner, even under the Notion of *Fluxions*, or *Differentials*, themſelves. The great Defect of both which Hypotheſes is, that they actually are not (as ſhall be effectually demonſtrated by and by) the Foundation of that for which they are taken. For a *Fluxion*, ſhould it be granted to carry ſome ſeeming Analogy to the true Foundation, yet in itſelf (See the Preface to this Work) it poſſibly can have no Exiſtence in Nature. And a *Differential*, if it have any Exiſtence in Nature, can poſſibly be nothing elſe, but ſuch an *Indiviſible* as is mentioned in the Beginning of the fourth Section: And, if ſo, then the *Indiviſible* ~~right~~ in the firſt Conſectary of that λ Section muſt be a *Differential.* From whence it neceſſarily follows, that the *Differential* of the circular Arch AVM $=$ na in the *Reſolutum* there is equal to n dividing the whole Series $z + \frac{1}{4} \times \frac{1}{6} \times z^3 + \frac{1}{4} \times \frac{1}{6} \times \frac{3}{8} \times \frac{3}{10} z^5 +$ &c. which is abſolutely falſe, according to the ſuppoſed Nature of a *Differential.* And thus the ingenious Inventors (whoever they were) of both theſe Hypotheſes, through this Ignorance of the true Principle, from whence their own Concluſions do neceſſarily proceed, have brought their Work under inextricable Difficulties: The former (to ſay no worſe) by making an unintelligible Myſtery of that, which in it ſelf is clear; and the latter, by reducing a certain Truth to the Neceſſity of a known Con~~dition.~~tradiction

In every Example of both the Direct and Inverſe Method, let it firſt of all be obſerved, that, before the ultimate Values of the Variants of any given Equation can be determined, the whole Equation on the ſame Side muſt always (Theo. 16.) be brought to the ſame Variant, by ſubſtituting each of its Equals to the other Variants, Theorem XXVIII.

S E C T. VIII.

Of the firſt Application of Ultimators.



LET VMN in the annexed Figure repreſent all Semi-curves univerſally whoſe right Axis is TQ. From which every right Line T S, being ſo drawn as to touch the Convexity of the Curve in the Point M without entring into it, is called a Tangent, to which the Perpendicular MR $=$ c, which again meets the Axis in R. is called the Normal. And to both theſe the Perpendicular PM to the Axis is called the Semiordinate, VP the Abſciſſa, TP the Subtangent, and PR the Subnormal. If then we always put $PM = s$, $VP = v$ and $VR = r$, we have $PR = r - v$; and becauſe $PR : PM = PM : TP$ we have $TP = \frac{ss}{r - v}$. Alſo, becauſe $\overline{PM}^2 + \overline{PR}^2$ equals $\overline{MR}^2$, in all Curves we have $ss + rr - 2rv + vv = cc$, or $2rv - vv - ss = rr - cc$. Theorem XXIX.

Hence it appears in all Curves with their concave Side to the right Axis T Q, that the greater Abſciſſa V P has always the greater Semiordinate P M; and each are the greateſt that they poſſibly can be to the ſame Arch V M, or to the ſame intercepted Axis V R. Therefore the Subnormal P R, and conſequently the Normal M R are each the leaſt that they poſſibly can be to the ſame Arch V M, or the ſame intercepted Axis V R. Therefore if in the laſt Equation c and r be invariable, we have $r^2 - c^2$ an *Ultimum*. Conſequently, the Ultimate of that Equation (Theo. 19.) is $2rv^\circ - 2vv^\circ - 2ss^\circ = 0$, or (dividing by 2) $ss^\circ = \overline{r - v} \times v^\circ$. Whence $v^\circ : s^\circ = s : r - v$. That is in all Curves, as the Ultimator of the Abſciſſa is to the Ultimator of the Semiordinate; ſo is the Semiordinate itſelf to the Subnormial. Or, which is all one, as the Ultimator of the Abſciſſa is to the Ultimator of the Semiordinate; ſo is the Subtangent to the Semiordinate, THEOREM XXX.

For *Example*. Suppoſe V M to be the Arch * of an Ellipſis, whoſe Parameter is 2 p, tranſverſe Diameter 2 d to the tranſverſe Vertex V. and conſequenrly its Equation is $2pv - \frac{p}{d}vv = ss$. Here Theo. 19.) $2pv^\circ - \frac{2p}{d}vv^\circ = 2ss^\circ$: whence $\frac{pd - pv}{ds}v^\circ = s^\circ$. Therefore (by the laſt) $v^\circ : \frac{pd - pv}{ds}v^\circ = s : PR$, and and $PR = p - \frac{p}{d}v = r - v$. Whence $v = d \times \frac{r - p}{d - p}$, and $r - v = p \times \frac{d - r}{d - p} = PR$.

Now in the Circle, we know that the Normal M R is always equal to the Semidiameter $c = r = VR$. Since therefore 1 is the Semidiameter of the Circle, whoſe Arch V M = U, and Semiordinate P M = S in the ſecond Conſectary of S the ſixth ection,

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* *See* Ellipſis *in the Appendix.*

tion, we muſt have $c\,U =$ the Arch V M here, and $1 : c = s : c\,S =$ the Semiordinate here. Whence if we put $u =$ the Arch V M here, we have $U = \frac{u}{c}$, and $S = \frac{s}{c}$. Which Values of U and S being every where ſubſtituted for them in the *Revolutum* of that Confectary, and both Sides multiplied by c, we have $VM = u$ here equal to the Series $s + \frac{1}{6c^2}s^3 + \frac{3}{40c^4}s^5 + \frac{5}{112c^6}s^7 + \frac{35}{1152c^8}s^9 + \frac{63}{2816c^{10}}s^{11} + \&c.$ Whoſe Ultimator (Theo. 19.) is s° multiplying the Series $1 + \frac{1}{2c^2}s^2 + \frac{3}{8c^4}s^4 + \frac{5}{16c^6}s^6 + \frac{35}{128c^8}s^8 + \frac{63}{256c^{10}}s^{10} + \frac{231}{1924c^{12}}s^{12} + \frac{429}{2048c^{14}}s^{14} + \frac{6435}{32768c^{16}}s^{16} + \&c.$ Which laſt Series is found to be equal to $\frac{cs^\circ}{\overline{cc - ss}|^{\frac{1}{2}}}$. Therefore the Ultimator of every circular Arch VM is equal to $\frac{cs^\circ}{\overline{cc - ss}|^{\frac{1}{2}}} = u^\circ$. And the Square of that Ultimator is $\frac{ccs^\circ s^\circ}{cc - ss} = u^\circ u^\circ$. Now from the Nature of the Circle we have $v = c - \overline{cc - ss}|^{\frac{1}{2}}$: Whence (Theo. 17, 21.) we have $v^\circ = \frac{ss^\circ}{\overline{cc - ss}|^{\frac{1}{2}}}$, and conſequently $v^\circ v^\circ = \frac{sss^\circ s^\circ}{cc - ss}$. But $s^\circ s^\circ + v^\circ v^\circ = s^\circ s^\circ + \frac{sss^\circ s^\circ}{cc - ss} = \frac{ccs^\circ s^\circ}{cc - ss}$. Therefore $s^\circ s^\circ + v^\circ v^\circ = u^\circ u^\circ$. That is, in the Circle the Sum of the Squares of the Ultimators of the Abſciſſa and Semiordinate is equal to the Square of the Ultimator of the Arch. But again, ſince the Ultimators s° and v° in the Equation $s^\circ s^\circ + v^\circ v^\circ = u^\circ u^\circ$ are connected with nothing particular to any Curve, but the Ultimator u° of the circular Arch $VM = u$; and ſince the Ultimator u° of the circular Arch $VM = u$ muſt infallibly have the ſame Ratio to the Ultimator of the Arch of every other Curve univerſally, which in the univerſal Equation $ss^\circ = \overline{r - v} \times v^\circ = PR \times v^\circ$ (Theo. 30.) the Ultimators of its Semiordinate and Abſciſſa have to the ſame univerſally, and which its Subnormal P R has to every other Subnormal; it follows therefore that the Equation $s^\circ s^\circ + v\ v = u^\circ u^\circ$ is univerſal. But to put this beyond all Doubt, we need only conſider, that in all aſſignable Curves whatever, the ſmaller Part we take of them, the nearer ſtill does that Part approach to an Equality with its reſpective Chord (ſee the Beginning of the ſixth Section): And in ſuch aſſignable Curves as do continually depart from their Axes, the farther alſo the Curve is produced from its Vertex, the nearer ſtill does the Curve itſelf approach to an Equality with its Chord. It therefore neceſſarily follows, that every aſſignable Curve in its ultimate State, being infinitely either ſhort or long, can be as nothing elſe but its Chord. Conſequently, in every Curve univerſally (ſince the ſame holds between its reſpective Abſciſſa, Semiordinate and Chord) The Sum of the Squares of the Ultimators of the Semiordinate and Abſciſſa is equal to the Square of the Ultimator of the Arch. Theorem XXXI.

Obſerve (Theo. 30.) that every Ultimator of the Abſciſſa is $v^\circ = \frac{ss^\circ}{r-v} = \frac{ss^\circ}{\overline{cc-ss}^{\frac{1}{2}}}$, and every Ultimator of the Semiordinate is $s^\circ = \frac{r-v}{s} v^\circ = \frac{r-v}{\overline{cc-\overline{r-v}^{2}}^{\frac{1}{2}}} v^\circ$. Theorem XXXII.

If therefore in the Equation (Theo. 31.) $s^\circ s^\circ + v^\circ v^\circ = u^\circ u^\circ$ we ſubſtitute $\frac{ss^\circ}{\overline{cc-ss}^{\frac{1}{2}}}$ for v°; we have $s^\circ s^\circ + \frac{s^2 s^\circ s^\circ}{cc-ss} = \frac{c^2 s^\circ s^\circ}{cc-ss} = u^\circ u^\circ$. Whence $\frac{cs^\circ}{\overline{cc-ss}^{\frac{1}{2}}} = u^\circ$. That is in every Algebraic Curve, as the Subnormal P R is to the Normal MR; ſo is the Ultimator of the Semiordinate to the Ultimator of the Curve. Theorem XXXIII.

Alſo, if in the Equation (Theo. 31.) $s\ \underline{s + v^\circ}_{2}\ v^\circ = u^\circ \times u^\circ$ we ſubſtitute $\frac{r-v}{\overline{cc-\overline{r-v}^{2}}^{\frac{1}{2}}} v^\circ$ for s°, we have. $\frac{\overline{r-v}^{2}}{cc-\overline{r-v}^{2}} v^\circ v^\circ + v^\circ$

$v^\circ = \frac{cc}{cc - \overline{r - v}^{2}}$ $v^\circ v^\circ = u^\circ u^\circ$. Whence $\frac{c}{\overline{cc - \overline{r - v}^{2}}^{\frac{1}{2}}} v^\circ =$ $\frac{c}{s} v^\circ = u^\circ$. That is, As the Semiordinate is to the Normal; ſo is the Ultimator of the Abſciſſa to the Ultimator of the Curve. THEOREM XXXIV.

But from the former of theſe two laſt Theorems, we have $\frac{cs^\circ}{\overline{cc - ss}^{\frac{1}{2}}} = u^\circ$ univerſally, juſt as in the Circle. Therefore, if from the particular Equation of any Curve, we find the Value of the Normal MR = c; the Length of the Arch muſt be $u = s + \frac{1}{6cc} s^3 + \frac{3}{40c^4} s^5 + \frac{5}{112c^6} s^7 + \frac{35}{1152c^8} s^9 + \frac{63}{2816c^{10}} s^{11} + \frac{231}{13312c^{12}} s^{13} + \frac{143}{10240c^{14}} s^{15} + \frac{6435}{557056c^{16}} s^{17} +$ &c. THEOREM XXXV.

Hence, becauſe in all ſimilar Figures, the correſpondent Sides are in the ſame direct Ratio to one another; therefore from the *Compoſitum* of Confect. 2. Sect. VI. from the given Length u of any Curve, and its given Normal c, we have $u - \frac{1}{2c} \times \frac{1}{3c} u^3 + \frac{1}{2c} \times \frac{1}{3c} \times \frac{1}{4c} \times \frac{1}{5c} u^5 - \frac{1}{2c} \times \frac{1}{3c} \times \frac{1}{4c} \times \frac{1}{5c} \times \frac{1}{6c} \times \frac{1}{7c} u^7 +$ &c. $= s$. THEOREM XXXVI.

Hence the Circle is as the Ultimate to all other Curves; and (if we pleaſe) we may carry them all to the ſame Exactneſs, as is done in the ſixth Section.

Let it now be required to determine the Subnormal PR to the common * Cycloid, whoſe generating Circle has the Radius VC = c, and their common Abſciſſa VP = v,

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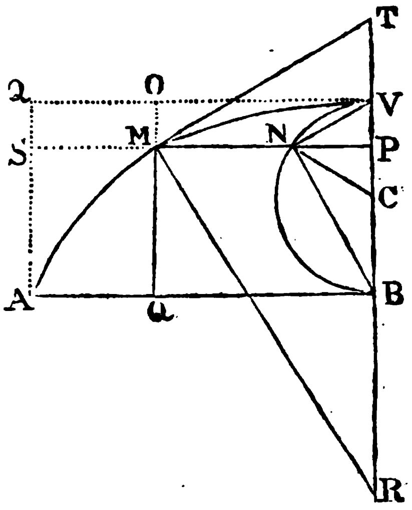
* *See* Cycloid *in the Appendix.*

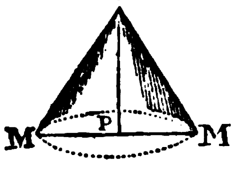
Put N P = s, and M N = u, which laſt by the known Property of the Cycloid is always equal to the Length of the correſpondent circular Arch VN. Whence the Semiordinate of the Cycloid is u+s. The Ultimator therefore of u, as the Arch V M of the Circle, being (Theo. 34.) $u^{\circ} = \frac{cs^{\circ}}{\overline{cc-ss}^{\frac{1}{2}}}$. Hence the Ultimator of the Semiordinate M P of the Cycloid muſt (Theo. 18.) be $\frac{cs^{\circ}}{\overline{cc-ss}^{\frac{1}{2}}} + s^{\circ} = \frac{cs^{\circ} + \overline{cc-ss}^{\frac{1}{2}}s^{\circ}}{\overline{cc-ss}^{\frac{1}{2}}}$. Alſo becauſe $VP = v = c - \overline{cc-ss}^{\frac{1}{2}}$; Therefore (Theo. 17.) the Ultimator of the Abſciſſa is $\frac{ss^{\circ}}{\overline{cc-ss}^{\frac{1}{2}}}$ (Theo. 33). Conſequently, (Theo. 30.) $\frac{ss^{\circ}}{\overline{cc-ss}^{\frac{1}{2}}} : \frac{cs^{\circ} + \overline{cc-ss}^{\frac{1}{2}}s^{\circ}}{\overline{cc-ss}^{\frac{1}{2}}}$, or (dividing by $\frac{s^{\circ}}{\overline{cc-ss}^{\frac{1}{2}}}$) $s : c + \overline{cc-ss}^{\frac{1}{2}} = u + s : \overline{u+s} \times \frac{c + \overline{cc-ss}^{\frac{1}{2}}}{s}$, = P R the Subnormal required. But $c + \overline{cc-ss}^{\frac{1}{2}} = BC + CP = PB$; therefore $\frac{MP \times PB}{NP} = PR$. Whence it follows that the Normal M R of the Cycloid muſt be always parallel ro the Chord N B of the Complement to the Semicircle of the Correſpondent circular Arch N V: or, which is all one, the Tangent M T of the Cycloid muſt be parallel to its correſpondent Chord N V in the Circle.



When the Ultimate of any one Part of a Subject plainly includes within itſelf the Ultimate of every other Part; then, all the other Parts being rejected, the Ultimate of that alone muſt determine the Queſtion. THEOREM XXXVII.

Example I. To determine the Dimenſions of ſuch a right Cone MVMP as ſhall have a ſolid Content equal to the given Cube b^3, within the greateſt Superficies, which a right Cone of the ſame ſolid Content poſſibly can have.

Let the Radius $MP = PM$ of the Cone equal a, and put $q =$ the Circumference of a Circle whoſe Radius is r. Whence, becauſe $r : q = a : \frac{qa}{r}$, we have $\frac{qa}{r}$ for the Circumference of the Baſe, and $\frac{qaa}{2r}$ its Area. And becauſe the Content of the right Cone is equal to the Product of the Area of its Baſe multiplied by one Third its Height, we have $\frac{1}{3} VP \times \frac{qaa}{2r} =$ bbb by the Queſtion. Whence $VP = \frac{6b^3r}{qaa} =$ the Cone's perpendicular Height. Whence again $\overline{\overline{MP}^2 + \overline{VP}^2}^{\frac{1}{2}} = \overline{\frac{36b^6r^2}{q^2a^4} + a^2}^{\frac{1}{2}} =$ MV, and MV multiplying $\frac{qa}{2r}$ half the Circumference of the Baſe gives $\overline{\frac{9b^6}{aa} + \frac{q^2a^4}{4rr}}^{\frac{1}{2}} =$ the convex Surface of the Cone, whoſe Ultimate of the Semidiameter $a = MP$, and conſequently that alſo of the Baſe. Therefore the Square alone of that convex Surface, *viz.* $\frac{9b^6}{aa} + \frac{q^2a^4}{4rr}$, whoſe Ultimate is $-\frac{18b^6}{aaa} a^{\circ} + \frac{q^2a^3}{rr} a^{\circ} = 0$ (Theo. 24, 19.) gives $a = b \times \overline{\frac{3r \times 2^{\frac{1}{2}}}{q}}^{\frac{1}{3}} =$ the Semidiameter of the Baſe.



From the laſt Theorem alſo we have the Method of determining the Ultimate of a Tangent, when the Curve is ſuch (ſee the two annexed Figures) as to have in it a Point M of contrary Flexion: That is, when the Curve on one Side of M has its Concavity, and on the other its Convexity, towards the given Axis. The former of which is called the *Inflexion*, and the latter the *Retrogreſſion* of the Curve, with Reſpect to that Axis. Thus in the former Figure the Curve begins with its Inflexion to the given Axis V R, and has all its Tangents, during that Flexion, falling between T and V, upon the ſame Axis produced. And in the other Figure the Curve begins with its Retrogreſſion to the given Axis VQ, and has all its Tangents, during that Flexion falling between T and V, upon another Axis produced from the ſame Point V perpendicular to the given Axis V Q. Upon which other Axis TR in the latter Figure, it is plain, that the Abſciſſa V S is the ſame with the Semiordinate P M upon the given Axis, and the Semiordinate S M the ſame with the Abſciſſa V P. Therefore, in all ſuch Curves as theſe the Ultimates of the Tangent, Subtangent, Normal, and Subnormal are included (Theo. 37.) within the Ultimate of the Portion T V of the produced Axis, which is intercepted between the Vertex V of the Curve, and the laſt Tangent T M of that Flexion. But T V=T P—V P. Since then (Theo. 30.) $PR = \frac{ss^\circ}{v^\circ}$, whence (Theo. 29.) $\frac{ss^\circ}{v^\circ} : s = s : \frac{sv^\circ}{s^\circ} = TP$, we have $TV = \frac{sv^\circ}{s^\circ} - v = \frac{sv^\circ - vs^\circ}{s^\circ}$. Put $v^\circ = a$, and $s^\circ = e$, whence $TV = \frac{sa - ve}{e}$, the Ultimator of which (Theo. 23.) is $\frac{sea^\circ + aes^\circ - eev^\circ - sae^\circ}{ee}$, or (becauſe $v^\circ = a$, and $s^\circ = e$) $s \times \frac{ea^\circ - ae^\circ}{ee}$. And, dividing by s, we have the Ultimator of $\frac{TV}{s}$ equal to $\frac{ea^\circ - ae^\circ}{ee}$, whoſe Subject (Theo. 23.) is $\frac{a}{e} = \frac{v^\circ}{s^\circ}$ Conſequently, in all Equations of Curves.

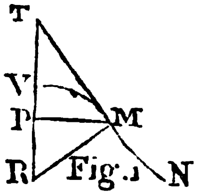


Fig. 1

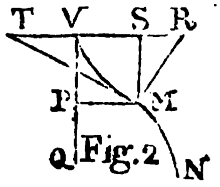


Fig. 2

Divide the Ultimator of the Abſciſſa by the Ultimator of the Semiordinate, and the Ultimate of that Quotient will determine the Length of each to the Point of the contrary Flexion. THEOREM XXXVIII.

Example 2. Let it be required to determine the Point M of contrary Flexion (Fig. 2.) to the Curve, whoſe Equation is $bv^2 = v^2 s + b^2 s$, putting b equal to any Quantity aſſignable.

Here we have $\frac{s^{o}}{v^{o}} = \frac{2b^3 v}{\overline{bb+vv}^{2}}$; where, as has been ſhewn, when a Curve begins with its Convexity towards the given Axis, the s becomes v, and the v becomes s to the Axis of the Tangent. Whence (Theo. 38, and 23.) we have $\frac{2b^5 v^{o} - 6b^3 v^2 v}{\overline{bb+vv}^{3}} = 0$. Therefore, (dividing by $\frac{2b^3 v^{o}}{\overline{bb+vv}^{3}}$) we have $b^2 - 3v^2 = 0$, or $v^2 = \frac{1}{3}b^2$, and $s = \frac{bvv}{bb+vv} = \frac{1}{4}b$.

Example 3. Suppoſe VMN (Fig. 1.) to be the Conchoid of *Nichomedes* in which putting b= the Diſtance (upon the given Axis) of the Aſymptote from the Center, p= the Diſtance (upon the ſame) of the Aſymptote from the Vertex, and $a = p - v$; let it be required to d termine a and s to the Point M of contrary Flexion.

Here the Equation of the Curve is $\frac{sa}{b+a} = \overline{pp-aa}^{\frac{1}{2}}$. Whence (Theo. 23, 21.) $\frac{\overline{b+a}\times as^{o} + bsa^{o}}{\overline{b+a}^{2}} = \frac{-aa^{o}}{\overline{pp-aa}^{\frac{1}{2}}}$ or (becauſe $s = \frac{\overline{b+a}\times\overline{pp-aa}^{\frac{1}{2}}}{a}$) $\frac{\overline{b+a}\times a s^{o} + \frac{b}{a}\times\overline{b+a}\times\overline{pp-aa}^{\frac{1}{2}} a^{o}}{\overline{b+a}^{2}} = \frac{as^{o} + \frac{b}{a}\times\overline{pp-aa}^{\frac{1}{2}} a^{o}}{b+a}$ $= \frac{-aa^{o}}{\overline{pp-aa}^{\frac{1}{2}}}$, or $a^2 s^{o} + b\times\overline{pp-aa}^{\frac{1}{2}} a^{o} = -\overline{b+a}\times\frac{a^2 a^{o}}{\overline{pp-aa}^{\frac{1}{2}}}$, or $\frac{s^{o}}{a^{o}} = \frac{-a^3 - bpp}{\overline{p^2 a^4 - a^6}^{\frac{1}{2}}}$, the Ultimate of which laſt (Theo. 38, 23.) is $\overline{2bp^2 - 3ba^2 - a^3}\times\frac{ppa^{1}a^{o}}{\overline{ppa^4 - a^6}^{\frac{3}{2}}} = 0$. Conſequently $a^3 + 3ba^2 = 2bpp$; or (becauſe $a = p - v$) $v^3 - 3\times\overline{b+p}\times v^2 + 3\times\overline{2b+p}\times pv = \overline{b+p}\times p^2$

Thus, as it before was taught (Theo. 30.) how to find the Ultimate of and Abſciſſa and its Semiordinate to any given Tangent; ſo theſe two laſt Examples determine the ſame (Theo. 38.) to the Ultimate of a Tangent. Which may ſuggeſt to us the Notion of Ultimators of Ultimators, to be denoted as follows. When any Ultimator as a° is conſider'd as a Subject, the Ultimator $a^{\overset{\circ}{\circ}}$ of that may be called a ſecond Ultimator; the Ultimator $a^{\overset{\circ}{\circ\circ}}$ of that again, a third Ultimator; the Ultimator $a^{\overset{\circ\circ}{\circ\circ}}$ of that again a fourth Ultimator, &c.

The firſt Ultimator therefore of a^n is $na^{n-1}a^{\circ} = na^{n-1}z$; the Ultimator of $na^{n-1}z$ (Theo. 22.) is $na^{n-1}z^{\circ} + n\times\overline{n-1}\times za^{n-2}a^{\circ} = na^{n-1}y + n\times\overline{n-1}\times a^{n-2}z^2$; the Ultimator of which (Theo. 22.) is $na^{n-1}y^{\circ} + n\times\overline{n-1}\times y\,a^{n-2}a^{\circ} + 2n\times\overline{n-1}\times a^{n-2}zz^{\circ} + n\times\overline{n-1}\times\overline{n-2}\times z^2a^{n-3}a^{\circ} = n\times a^{n-1}x + 3n\times\overline{n-1}\times a^{n-2}zy + n\times\overline{n-1}\times\overline{n-2}\times a^{n-3}z^3$; the Ultimator of which again (Theo. 22.) is $n\times a^{n-1}x^{\circ} + n\times\overline{n-1}\times a^{n-2}a^{\circ}x + 3n\times\overline{n-1}\times\overline{n-2}\times a^{n-3}a^{\circ}zy + 3n\times\overline{n-1}\times a^{n-2}zy^{\circ} + 3n\times\overline{n-1}\times a^{n-2}yz^{\circ} + 3n\times\overline{n-1}\times\overline{n-2}\times a^{n-3}z^2z^{\circ} + n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times a^{n-4}a^{\circ}z^3 = na^{n-1}u + 4n\times\overline{n-1}\times a^{n-2}zx + 6n\times\overline{n-1}\times\overline{n-2}\,a^{n-3}z^2y + 3\times\overline{n-1}\times a^{n-2}yy + n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times a^{n-4}z^4$ Conſequently (reſtoring a° for z, $a^{\overset{\circ}{\circ}}$ for $y = z^{\circ}$, $a^{\overset{\circ}{\circ\circ}}$ for $x = y^{\circ} = z^{\overset{\circ}{\circ}}$, &c.) from the Subject a^n the firſt Ultimator is $na^{n-1}a^{\circ}$: The ſecond Ultimator is $na^{n-1}a^{\overset{\circ}{\circ}} + n\times\overline{n-1}\times a^{n-2}a^{\circ}a^{\circ}$: The third Ultimator is $n\times a^{n-1}a^{\overset{\circ}{\circ\circ}} + 3n\times\overline{n-1}\times a^{n-2}a^{\circ}a^{\overset{\circ}{\circ}} + \overline{n-1}\times\overline{n-2}$ /n $\times a^{n-3}a^{\circ}a^{\circ}a^{\circ}$: The fourth Ultimator is $na^{n-1}a^{\overset{\circ\circ}{\circ\circ}} + 4n\times\overline{n-1}\,a^{n-2}a^{\circ}a^{\overset{\circ}{\circ\circ}} + 6n\times\overline{n-1}\times\overline{n-2}\times a^{n-3}a^{\circ}a^{\circ}a^{\overset{\circ}{\circ}} + 3n\times\overline{n-1}\times a^{n-2}a^{\overset{\circ}{\circ}}a^{\overset{\circ}{\circ}} + n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times a^{n-4}a^{\circ}a^{\circ}a^{\circ}a^{\circ}$, &c. THEOREM XXXIX.

After the ſame Manner, the Subject $a^n e^m$ has the firſt Ultimator $ma^ne^{m-1}e^{\circ} + ne^ma^{n-1}a^{\circ}$; the ſecond Ultimator $ma^ne^{m-1}e^{\overset{\circ}{\circ}} + ne^ma^{n-1}a^{\overset{\circ}{\circ}} + 2mna^{n-1}e^{n-1}a^{\circ}e^{\circ} + m\times\overline{m-1}\,a^ne^{m-2}e^{\circ}e^{\circ} + n\times\overline{n-1}\,e^ma^{n-2}a^{\circ}a^{\circ}$; &c. THEOREM. XL.

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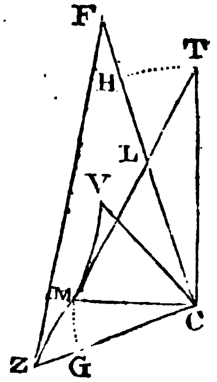
So again the Subject $\frac{a^n}{e^m}$ (Theo. 23.) has the firſt Ultimator $\frac{ne\,a^{n-1}\,a^{\circ} - ma^n\,e^{\circ}}{e^{m+1}}$, the ſecond Ultimator $\frac{ne^2a^{n-1}a^{\overset{\circ}{\circ}} - 2mnea^{n-1}a^{\circ}e^{\circ}}{e^{m+2}} + \frac{n\times\overline{n-1}\,e^2a^{n-1}a^{\circ}a^{\circ} + m\times\overline{m+1}\,a^ne^{\circ}e^{\circ}}{e^{m+2}}$ $\frac{-mea^ne^{\overset{\circ}{\circ}}}{e^{m+2}}$, &c. THEOREM. XLI.

In which three laſt Theorems, obſerve that when you ſubſtitute the given Numbers for n and m, that Quantity, whoſe Exponent by ſuch means becomes equal to nothing, muſt always have Unity ſubſtituted for it, and not a° or e°. And if any of the Coefficients become equal to nothing, that Term vaniſhes.

But to facilitate the Work, where we have Occaſion for theſe ſecond, third, &c. Ultimators, we need but conſider, that inevery Equation of different Ultimators we may aſſume any one of them at Pleaſure as the Ultimator of the laſt Variation, which its reſpective Quantity can admit of, according to the Conditions of the Queſtion. And if ſuch a one be the Ultimator of the laſt Variation of its Quantity, it is evident, that in itſelf it is henceforward as an invariable Quantity, and muſt continue the ſame through all other poſſible Variations of the other variable Quantities. And as ſuch it may be diſtinguiſhed by a Point over its Head. Thus if *x* be a variable Quantity, then let $\dot{x}^{\circ}$ be the invariable of its firſt Ultimator x°, $\dot{x}^{\overset{\circ}{\circ}}$ the invariable of its ſecond Ultimator $x^{\overset{\circ}{\circ}}$, &c. For Example, if a variable Quantity be expreſs'd by $\frac{x^{\circ}y}{y^{\circ}} = \frac{\dot{x}^{\circ}y}{y^{\circ}}$; then the Ultimator of that Quantity (Theo. 23.) will be $\frac{\dot{x}^{\circ}y^{\circ}y^{\circ} - \dot{x}^{\circ}yy^{\overset{\circ}{\circ}}}{y^{\circ}y^{\circ}} = \frac{x^{\circ}y^{\circ}y^{\circ} - yx^{\circ}y^{\overset{\circ}{\circ}}}{y^{\circ}y^{\circ}}$. Whenever therefore an Ultimator is to be thus ſuppoſed invariable, we muſt for Convenience Sake, pitch upon that (ſince it is at our Diſcretion to pitch upon which one we pleaſe,) which may bring the Equation to be expreſſed in the eaſieſt Manner for Reſolution, by depreſſing ſome Terms, and exterminating others. THEOREM. XLII.

But let it always be taken Notice, that the Point (.) here is uſed in a quite different Senſe from the ſame in Fluxions. And its Uſe will be farther ſhewn, when we come to the fifty-ſixth and fifty-ſeventh Theorems.

One Uſe of this Theorem will appear in determining the Point of contrary Flexion to every Spiral Curve, *i. e.* ſuch as continually varies its Axis about the Center C, having every Complement V C, M C, &c. to its Ordinates, terminating in the Point C, till at laſt ſuch Complement becomes equal to nothing in C. Suppoſe V M to be the Portion of ſuch a Spiral, which has its Convexity always towards the Center C, till it come to the Point M, from whence it begins to have its Concavity towards the Center C. Let M Z repreſent an infinitely ſmall Part of that Concavity on the contrary Side to V. Then let T M be a Tangent to the Curve in the Point M, meeting C T the Perpendicular to C M in T; and let F Z be a Tangent to the Curve in the Point Z, meeting C F the Perpendicular to C Z in F. Alſo let M G be the Arch of a Circle with the Radius M C upon the Center C, and let T H be the Arch of a Circle with the Radius T C upon the ſame Center.



Now Things being thus prepared, it is evident, if the Arch M Z be infinitely little, that the Tangent F Z, and Right Line L M muſt both coincide with the Tangent M T, and at the ſame Time the Subtangent C F and the Right-Line C L muſt both coincide with the Subtangent C T. Conſequently in that Caſe, we have $FH + HL = FL = 0$. And ſince M Z (Theo. 31.) is the Ultimator of the Curve; it follows, that M G muſt be the Ultimator of the Correſpondent circular Arch, whoſe Radius is $CM = y$, and $ZG = y^\circ$ muſt be the correſpondent Ultimator of C M, and F H the correſpondent Ultimator of the Subtangent C T.

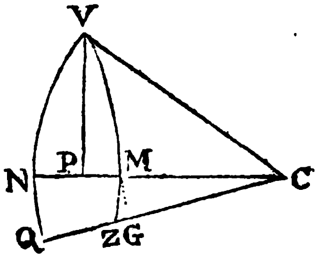
Put $MG = x^\circ$, and (Theo. 42.) make x° invariable; whence $\dot{MG} = \dot{x}^\circ$. Then, becauſe the Angles M C T and Z C F are both right, and $MCZ = HCT$, and in the Circle every Radius $CG = CM$ is perpendicular to the included Arch M G: There-

ſore

fore the Angle MCT = ZGM. Again, becauſe of the infinite Propinquity of CZ to CM, and of ZT to ZF, we have the Angle MCZ = FZT = o; and in the Triangle MCZ the external Angle TMC equals the Sum of the two internal ones MZC + MCZ, therefore TMC = MZC = MZG. Conſequently from the Similarity of the Triangles MZG and TMC, we have ZG : MG = MC : TC, or $y^{\bullet} : \dot{x}^{\circ} = y : \frac{\dot{x}^{\circ} y}{y}$ = TC. Alſo from the Similarity of the Iſoſceles Triangles CMG and CTH we have CM : MG = CT : GH, or $y : \dot{x}^{\circ} = \frac{\dot{x}^{\circ} y}{y^{\circ}} : \frac{\dot{x}^{\circ}\dot{x}^{\circ}}{y^{\circ}}$ = TH. Laſtly, becauſe the Angle LCT = MCN = o, therefore MLC = LTC = TLF. But the Angle THL (for the Reaſon above) is right; whence the Triangles TMC and LTH are ſimilar, and CM : CT = TH : HL, or $y : \frac{\dot{x}^{\bullet} y}{y^{\bullet}} = \frac{\ddot{x}x}{y^{\bullet}} : \frac{\dot{x}^{\bullet}\dot{x}^{\circ}\dot{x}^{\bullet}}{y^{\circ}\,\dot{y}^{\bullet}_{\circ}} :$ = HL. Then FH being the Ultimator of TC = HC = $\frac{x,}{y^{\bullet}}$ muſt (Theo. 41.) give $\frac{\dot{x}^{\bullet}\dot{x}^{\circ}\, y^{\circ} y^{\circ} - \dot{x}^{\circ}\, v\, \overset{\circ}{y}^{\bullet}}{y^{\circ} y^{\bullet}}$ = FH. Conſequently, cauſe FH + HL = o, we have $\frac{\ddot{x}^{\circ} y^{\bullet} y^{\circ} - \dot{x}^{\circ} y^{\circ} \overset{\bullet}{y}^{\circ} + \dot{x}^{\circ}\;\dot{x}^{\circ}\dot{x}^{\circ}}{y^{\circ}\, y^{\circ}}$ = o. Whence $y^{\circ} y^{\circ} + x^{\bullet} x^{\circ} = y^{\bullet} y^{\circ} + x^{\bullet} x^{\circ} = y^{\circ} \overset{\bullet}{y}^{\circ}$. THEOREM XLIII.

For *Example*, Let it be required to determine the Point M of contrary Flexion to the Parabolic Spiral, which is generated by the ſame Law with a common Parabola, whoſe Parameter is 2 p, having the Circumference VNQ &c. of a Circle for its Axis; and conſequently all its Semiordinates NM, QZ, &c. with their reſpective Complements MC, ZC are each together equal to VC the Radius of that Circle, and do all meet in the Center C.

Here putting the given Radius VC = NC = c, VN = u NM = z, MC = GC = y = c − z, and the ſuppoſed infinitely ſmall Arch MG = x° = x° (as in the two laſt Theorems) invariable; whence NQ $= u^{\circ}$, and ZG $= y^{\circ}$: then the Equation of the Curve will be $2pu = zz = c^2 - 2cy + y^2$; whence $pu^{\circ} = -cy^{\circ} + yy^{\circ}$. And becauſe NC : NQ = MC : MG, or $c : u^{\circ} = y : \dot{x}^{\circ}$, we have $yu^{\circ} = c\dot{x}^{\circ}$. From which two given Equations we have $u^{\circ} = \frac{y-c}{p} y^{\circ} = c\frac{\dot{x}^{\circ}}{y}$. Whence $\frac{y^2 - cy}{pc} y^{\circ} = \dot{x}^{\circ}$, and $\frac{y^4 - 2cy^3 + c^2y^2}{ppcc}$

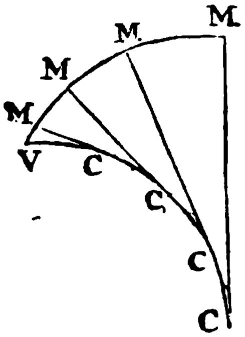


$y^{\circ}y^{\circ} = \dot{x}^{\circ}\dot{x}^{\circ}$. But the Ultimate Value of MC $= y$ from the Equation $\frac{y^2y^{\circ} - cyy^{\circ}}{pc} = \dot{x}^{\circ}$ gives $2yy^{\circ}y^{\circ} + y^2y^{\overset{\circ}{\circ}} - cyy^{\overset{\circ}{\circ}} - \cancel{cyy^{\circ}} - cy^{\circ}y^{\circ} = 0$; whence $yy^{\overset{\circ}{\circ}} = \frac{c - 2y}{y - c} y^{\circ}y^{\circ}$. Conſequently, theſe Values of $\dot{x}^{\circ}\dot{x}^{\circ}$, and $yy^{\overset{\circ}{\circ}}$ being ſubſtituted in the laſt Theorem give $\frac{p^2c^2 + y^4 - 2cy^3 + c^2y^2}{ppcc} y^{\circ}y^{\circ} = \frac{c - 2y}{y - c} y^{\circ}y^{\circ}$. Whence we have $y^5 - 3cy^4 + 3c^2y^3 - c^3y^2 + 3c^2p^2y = 2c^3p^2$, (or becauſe $y = c - z$) $z^5 - 2cz^4 + c^2z^3 + 3c^2p^2z = c^3p^2$.

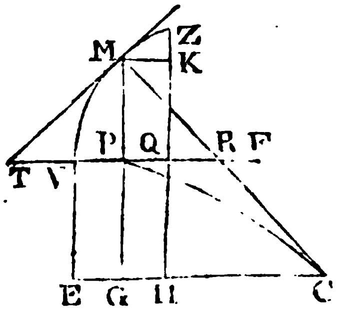
Another Uſe of Theo. XLII. is in determining the Radius of an involute Curve, the Meaning of which is as follows:

Suppoſe the Curve V C C C C have an Infinity of Tangents drawn to it in the Points C, C, C, &c. till we come to the Vertex V, and the Length of each of the Arches V C, V C, V C, &c. be laid upon its reſpective Tangent, from C to M; then the Curve formed from the Point V through the Extremities of theſe Tangents M, M, M, &c. is called an Involute with reſpect to the Curve V C C C C, which is therefore called the Evolute. And each of theſe Tangents MC (which is always equal to the Length of the reſpective Arch V C of its Evolute) is called the Radius of the reſpective Arch of the Involute. Hence

Every Involute is compoſed of an Infinity of infinitely little Arches of Circles, whoſe Radii are ſucceſſively equal to C M, C M, C M, and the Centers are the reſpective Points C, C, C, &c. in the Evolute. Conſequently, if we begin at the Point V, the Curvature of the Involute muſt always decreaſe from thence, and approach nearer to a Right Line in Proportion as its Radius encreaſes. Conſequently every Radius falls perpendicularly upon the Involute. THEOREM XLIV.



Let V F in the following Figure be the Axis of the Involute V M N, whoſe right Semi-ordinate is $MP = s$, Abſciſſa $VP = v$, Arch $VM = u$, Radius to its Evolute $MC = q$, to which T M is the Tangent: and put $MG = z$, to which Z H is parallel to V F. Then ſuppoſing (Theo. 31.) M Z to be infinitely little, we have $MK = PZ = GH = v^{\circ}$, $ZK = s^{\circ}$, and $MZ = u^{\circ} = \overline{v^{\circ} v^{\circ} + s^{\circ} s^{\circ}}^{\frac{1}{2}}$; and becauſe (by Reaſon of ſimilar Triagales) $MK : MZ = TP : TM = MG : MC$, or $v^{\circ} : \overline{v^{\circ} v^{\circ} + s^{\circ} s^{\circ}}^{\frac{1}{2}} = z : \frac{z}{v^{\circ}} \times \overline{v^{\circ} v^{\circ} + s^{\circ} s^{\circ}}^{\frac{1}{2}} = q$. Whence (Theo. 42.) putting $v^{\circ} = \dot{v}^{\circ}$, we have



$$\frac{z\dot{v}^{\circ}\dot{v}^{\circ}z^{\circ} + zz^{\circ}s^{\circ}s^{\circ} + z^{2}s^{\circ}s^{\circ\circ}}{\dot{v}^{\circ} \times \overline{\dot{v}^{\circ}\dot{v}^{\circ} + s^{\circ}s^{\circ}}^{\frac{1}{2}}} = 0, \text{ or } z = \frac{v^{\circ}v^{\circ}z^{\circ} + z^{\circ}s^{\circ}s^{\circ}}{-s^{\circ}s^{\circ\circ}} = MG.$$

Or, becauſe P G is invariable, and $PG + s = z$; whence (by Theo. 17.) $0 + s^{\circ} = z^{\circ}$, we have $z = \frac{v^{\circ}v^{\circ} + s^{\circ}s^{\circ}}{-s^{\circ\circ}} = MG$. Conſequently, ſubſtituting this for z in $MC = \frac{z}{v^{o}} \times \overline{v^{o} v^{o} + s^{o} s^{o}}^{\frac{1}{2}}$, we have $MC = \frac{\overline{v^{o}v^{o} + s^{o}s^{o}}^{\frac{3}{2}}}{-v^{o}s^{oo}} = q$.

THEOREM XLV.

For *Example*, let it be required to determine the Radius MC of the common Hyperbola taken as an Involute.

Here putting $2p =$ the Parameter, and $2d =$ the tranſverſe Diameter, and s and v as before; the Equation of the Hyperbola is $2dpv + pvv = dss$. Or putting $y = d + v$; whence (Theo. 17.) $y^{o} = v^{o}$, and $v = y - d$, the Equation becomes $py^{2} - pd^{2} = ds^{2}$. Whence $\overline{\frac{p}{d} y^{2} - pd}^{\frac{1}{2}} = s$, and $\frac{\frac{p}{d} y y^{o}}{\overline{\frac{p}{d} y^{2} - pd}^{\frac{1}{2}}} = s^{o}$: and making $y^{o} = v^{o}$, (as in the laſt Theorem) invariable, we have $\frac{-pp y^{o} y^{o}}{\overline{\frac{p}{d} yy - pd}^{\frac{1}{2}}} = \frac{-p^{2} y^{o} y^{o}}{s^{3}} = \overset{o}{s}^{o}$. And becauſe $\frac{\frac{p}{d} y y^{o}}{s} = s^{o}$, whence $\frac{ppyy}{ddss} y^{o} y^{o} = s^{o} s^{o}$; alſo becauſe $y^{o} y^{o} = v^{o} v^{o}$; therefore ſubſtituting theſe Values in the laſt Theorem $\frac{v^{o} v^{o} + s^{o} s^{o}}{-\dot{s}^{o}} = MG$, we have $\frac{s^{3} + \frac{pp}{dd} s y^{2}}{pp} = MG$. But (Theo. 30.) $PR = \frac{py}{d}$; whence $\overline{MR}^{2} = \overline{MP}^{2} + \overline{PR}^{2} = ss + \frac{pp}{dd} yy$: Conſequently $MG = \frac{s}{pp} \times \sqrt{ss + \frac{pp}{dd} yy}$ equals $\frac{\overline{MR}^{2} \times MP}{pp}$. And becauſe $MP : PR = MG : GC$, or $s : \frac{py}{d} = \frac{s}{pp} \times \sqrt{ss + \frac{pp}{dd} yy}$:

$\frac{1}{pp}\times\frac{py}{d}\times\sqrt{ss+\frac{pp}{dd}yy}$ we have $GC=\frac{\overline{MR}^2\times PR}{pp}$: Alſo, becauſe $PR:MR=GC:MC$, or $PR:MR=\frac{\overline{MR}^2 PR}{pp}=MC$, therefore $MC=\frac{\overline{MR}^3}{pp}=$ the Radius of the Involute required.

Hence, to find the Equation to the Evolute V C of the Hyperla V M, put $VE=PG=x=$ the Abſciſſa of the Evolute, and $EC=VP+GC=z$ its Semiordinate. But from the Equation of the Involute, we have $yy=\frac{ds^2+d^2p}{p}$, and $ss=\frac{py^2-pd^2}{d}$. Therefore in the Evolute, we have $VE=PG=MG-MP=\frac{s^3+\frac{pp}{dd}syy-pps}{pp}=\frac{\overline{d+p}\times s^3}{dpp}=x$, and $EC=VP+GC=y-d+\frac{y}{pd}ss+\frac{p}{d^3}y^3=\frac{\overline{d+p}\times y^3-d^4}{ddd}=z$. Whence $s^3=\frac{dppx}{d+p}$, and $y^3=\frac{d^3z+d^4}{d+p}$. Conſequently $y=d\times\overline{\frac{z+d}{d+p}}^{\frac{1}{3}}$ and $s=\overline{\frac{dppx}{d+p}}^{\frac{1}{3}}$. Which two Values of y and s being ſubſtituted in the Equation $pyy-ddp=dss$ give $pd\times\overline{\frac{z+d}{d+p}}^{\frac{2}{3}}-dp=\overline{\frac{dppx}{d+p}}^{\frac{2}{3}}$ the Equation of the Evolute. Whence putting $z+d=a$, and dividing all by p, we have $d\times\overline{\frac{aa}{\overline{d+p}^2}}^{\frac{1}{3}}-d=\overline{\frac{ddpxx}{\overline{d+p}^2}}^{\frac{1}{3}}$; and the Multiplication of all again by $\overline{d+p}^{\frac{2}{3}}$ gives

d ×

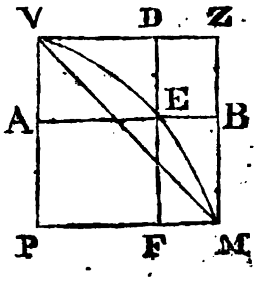
$d \times \overline{a}^{\frac{2}{3}} - d \times \overline{d+p}^{\frac{2}{3}} = \overline{d\,d\,p\,x\,x}^{\frac{1}{3}} = p^{\frac{1}{3}} \times \overline{d\,d\,x\,x}^{\frac{1}{3}}$, and dividing all again by d the Equation of the Evolute becomes $a^{\frac{2}{3}} - \overline{d+p}^{\frac{2}{3}} = \left.\frac{p}{d}\right|^{\frac{1}{3}} \times x^{\frac{2}{3}}$, or $\overline{a\,a}^{\frac{1}{3}} - \overline{d\,d+2\,d\,p+p\,p}^{\frac{1}{3}} = \left.\frac{p\,x\,x}{d}\right|^{\frac{1}{3}} = \left.\frac{p}{d}\right|^{\frac{1}{3}} \times \overline{x\,x}^{\frac{1}{3}}$ Or laſtly putting $\overline{z+d}^{\frac{2}{3}} = \overline{a\,a}^{\frac{1}{3}} = e^{1}$, and $\overline{x\,x}^{\frac{1}{3}} = u^{1}$, the Equation is $e - \overline{d+p}^{\frac{2}{3}} = \left.\frac{p}{d}\right|^{\frac{1}{3}} u$.

SECT. IX.

The Menſuration of Curve Surfaces and Solids.

IN all Curves (ſtill keeping $V P = v$, $P M = s$, and $VEM = u$) let $y =$ the Chord V M, which ſubtends the V E M in the annexed Figure; whence $v^2 + s^2 = y^2$. Then, becauſe it is plain that the Ultimator of the right angled Triangle M V P muſt be ſimular to the Triangle itſelf. Therefore, as the Semiordinate s is to the Chord y; ſo is the Ultimator s° to the Ultimator y°. Conſequently $s\,y^{\circ} = y\,s^{\circ}$. But it has been ſhewn that $y^{\circ}\,y^{\circ} = v^{\circ}\,v^{\circ} + s^{\circ}s^{\circ} = u^{\circ}\,u^{\circ}$: Therefore in their Ultimators $s\,y^{\circ}$ is the ſame with $s\,u^{\circ}$.

Let us now ſuppoſe the Arch $V E M = u$ to be turned round the Axis V P: and it is evident that the Ultimator of the Curve Superficies ſo formed muſt be the ſame with that of the Convex Superficies of the right Cone, which is at the ſame Time formed by the Chord $V M = y$. But if we put $b = 2\,c$ equal to the Diameter of a Circle, whoſe Circumference is q, then the Circumference of the Baſe of that Cone muſt be $\frac{2\,q\,s}{b}$ or $\frac{q}{c}\,s$, and the convex Superficies of the Cone muſt be $\frac{q}{2\,c}\,s\,y$, the Ultimator of which



laſt is $\frac{q}{2c} \times \overline{s\dot{y} + y\dot{s}}$, or becauſe (as has been ſhewn, $s\dot{y} = y\dot{s} = s\dot{u}$) $\frac{qs}{c}\dot{u} = \frac{qs}{c} \times \overline{\dot{v}\dot{v} + \dot{s}\dot{s}}$. That is, as the Radius c of any Circle, whoſe Circumference is q, is to the Product q s of that Circumference multiplied by the Semiordinate : ſo is the Ultimator $\dot{u}$ of the Curve, to the Ultimator $\frac{q}{c} s\dot{u}$ of the curve Superficies which is formed by the Rotation of the Curve V E M about the Abſciſſa V P. THEOREM XLVI.

Thus in the Circle, whoſe Radius is c, where $s^2 = 2cv - vv$. (Whence $\dot{s} = \frac{c - v}{\overline{2cv - vv}^{\frac{1}{2}}} \dot{v}$, and $\dot{s}\dot{s} = \frac{cc - 2cv + vv}{2cv - vv} \dot{v}\dot{v}$) we have $\frac{qs}{c} \times \overline{\dot{v}\dot{v} + \dot{s}\dot{s}} = \frac{q}{c} \frac{\times \overline{2cv - vv}^{\frac{1}{2}}}{\overline{2cv - vv}^{\frac{1}{2}}} \overline{\times cc\dot{v}\dot{v}}^{\frac{1}{2}} = q \quad \dot{v}$. Whoſe Subject q v is the curve Superficies about the Abſciſſa V P.

Hence, becauſe the Ultimator of every Curve, whoſe Semiordinate is s, and Normal to the Point M is c, when multiplied by s, makes $s\dot{s} + \frac{s^3\dot{s}}{2c^2} + \frac{3s^5\dot{s}}{8c^4} + \frac{5s^7\dot{s}}{16c^6} + \frac{35s^9\dot{s}}{128c^8} + \frac{63s^{11}\dot{s}}{256c^{10}}$. Therefore every ſuch curve Superficies muſt be equal to $\frac{qss}{c}$ multiplying the Series $\frac{1}{2} + \frac{s^2}{8c^2} + \frac{3s^4}{48c^4} + \frac{5s^6}{128c^6} + \frac{35s^8}{1280c^8} + \frac{21s^{10}}{1024c^{10}} +$ &c. THEOREM XLVII.

Hence again, it will be eaſy to conceive, that if we ſubſtitute v for s (Theo. 46.) we have a like Proportion for finding the Convex Superficies, which is formed by the Rotation of any Curve V E M about the Axis P M = s, *viz.* As the Radius c of any Circle, whoſe Circumference is q, is to the Product q v of that Circumference multiplied by the Abſciſſa; ſo is the Ultimator $\dot{u}$ of the Curve,

Curve, to the Ultimator $\frac{q}{c} v u^{\circ} = \frac{qv}{c} \times \overline{v^{\circ} v^{\circ} + s^{\circ} s^{\circ}}$ of the Superficies required. THEOREM XVIII.

Next to raiſe Theorems for finding the Ultimators of the Plane Areas and Solid Contents included within all Sorts of Curves, and Curve Surfaces formed from thence, ſomething muſt be premiſed by way of *Lemmata*, as the Foundation upon which the whole Superſtructure is to reſt.

Suppoſe then a Series of natural Numbers 1, 2, 3, 4, 5, &c. to proceed from Unity, till the Number of the Terms be n; and $A =$ the Sum of their firſt Powers $1 + 2 + 3 + 4 + 5$ &c. to $+ n$; $B =$ the Sum of their Squares $1 + 4 + 9 + 16 + 25$ &c. to n^2; $C =$ the Sum of their Cubes $1 + 8 + 27 + 64 + 125$ &c. to $+ n^3$; $D =$ the Sum of their fourth Powers to n^4; $E =$ the Sum of their fifth Powers to n^5; $F =$ the Sum of their ſixth Power to n^6; and univerſally $Z =$ the Sum of their m Powers to n^m. Then if the Series be increaſed to one Term more: that is, if we make the laſt Term and the Number of Terms equal to $n + 1$, it is plain, that the Sum of all the firſt Powers ſo increaſed will be $A + n + 1$, of all the Squares $B + n^2 + 2n + 1$, of all the Cubes $C + n^3 + 3n^2 + 3n + 1$, of all the fourth Powers $D + n^4 + 4n^3 + 6n^2 + 4n + 1$, of all the fifth Powers $E + n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1$, &c. And the Sum of all the m Powers will be $Z + \overline{n + 1}^{m}$, or (putting m for the n in the third Section, n for c, 1 for a, and the p, q, r, s, t, &c. there accordingly) $Z + n^m + m n^{m-1} + m p n^{m-2} + m p q n^{m-3} + m p q r n^{m-4} + m p q r s n^{m-5} +$ &c. But the Sum of all the m Powers of the $n + 1$ Number of ſuch Terms is had by the Addition of their Equals to $\overline{1 + n}^{m}$, as follows:

$$1^m = 1 + 0 + 0 + 0 + 0 + 0 + \&c.$$

$$\overline{1+1}^m = 1 + m + mp + mpq + mpqr + mpqrs + \&c.$$

$$\overline{1+2}^m = 1 + 2m + 4mp + 8mpq + 16mpqr + 32mpqrs + \&c.$$

$$\overline{1+3}^m = 1 + 3m + 9mp + 27mpq + 81mpqr + 243mpqrs + \&c.$$

$$\overline{1+4}^m = 1 + 4m + 16mp + 64mpq + 256mpqr + 1024mpqrs + \&c.$$

&c. &c. &c. &c. &c. &c. &c.

In ſumming up of which Serieſes, it is plain that the Sum of all the firſt Terms or Units is $n+1$; the Sum of all the ſecond Terms m is Am; the Sum of all the third Terms mp is Bmp; the Sum of all the fourth Terms mpq is $Cmpq$; the Sum of all the fifth Terms $mpqr$ is $Dmpqr$; the Sum of all the ſixth Terms $mpqrs$, is $Empqrs$, &c. Whence we have this univerſal Equation $n + 1 + Am + Bmp + Cmpq + Dmpqr + Empqrs +$, &c. $= Z + n^m + mn^{m-1} + mpn^{m-2} + mpqn^{m-3} + mpqrn^{m-4} + mpqrsn^{m-5} +$, &c. Conſequently, by making $Z = A$, and $m = 1$ for the firſt Powers, $Z = B$ and $m = 2$ for the Squares, $Z = C$ and $m = 3$ for the Cubes, $Z = D$ and $m = 4$ for the fourth Powers, &c. we have

$$\overline{n+1}^2 - \overline{n+1} = 2A$$

$$\overline{n+1}^3 - \overline{n+1} = 3A + 3B$$

$$\overline{n+1}^4 - \overline{n+1} = 4A + 6B + 4C$$

$$\overline{n+1}^5 - \overline{n+1} = 5A + 10B + 10C + 5D$$

$$\overline{n+1}^6 - \overline{n+1} = 6A + 15B + 20C + 15D + 6E$$

$$\overline{n+1}^7 - \overline{n+1} = 7A + 21B + 35C + 35D + 21E + 7F$$

$$\overline{n+1}^8 - \overline{n+1} = 8A + 28B + 56C + 70D + 56E + 28F + 8G$$

$$\overline{n+1}^9 - \overline{n+1} = 9A + 36B + 84C + 126D + 126E + 84F + 36G + 9H.$$

&c. &c. &c. &c.

Hence in the Series 1, 2, 3, 4, 5, &c. proceeding to the Number of Terms n. The Sum of all the

Firſt Powers $A = \frac{1}{2}n^2 + \frac{1}{2}n$.

Second Powers $B = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$.

Third Powers $C = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n$.

Fourth Powers $D = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{3}n$.

Fifth Powers $E = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$. **Theo. XLIX.**

Hence if any aſſignable Quantity n be ſuppoſed to be divided into an Infinity of equal Parts, the leaſt of which is equal to Unity, and it be required to find the Sum Z of all the m Powers to every Number of theſe Parts 1, 2, 3, 4, 5, 6, &c. to n: in this Caſe it is plain, (See the Beginning of Sect. VI.) that not only the Unit one, but alſo every lower Power of n, and every Sum of them made up of any finite Number of A', B', C', D', &c. are all taken together as nothing with Reſpect to the higheſt Power n^m itſelf. Conſequently in this Caſe we muſt univerſally have $\overline{m+1} \times Z = n^{m+1}$. Whence $Z = \frac{1}{m+1} n^{m+1}$. That is the Sum Z of an Infinity *n* of m Powers, whoſe Roots are 1, 2, 3, 4, 5, 6, &c. till we come to n the greateſt is equal to $\frac{1}{m+1}$ Times the $\overline{m+1}$ Power of that greateſt Term n. **Theorem L.**

Upon which laſt Theorem the Foundation is laid for determining the Ultimators to all Planes and Solids whatever, by the known Method of (Sect IV.) Indiviſibles. For by this Time it may ſufficiently appear, that in every greater Area V M P within the ſame Rectangle V Z M P. (See the laſt Figure) the Abſciſſa V P = v is higher in Power than the Semiordinate P M = s; and convertibly in every leſſer Area V M Z (making up rhe ſame Rectangle V Z M P) the Line V Z = s taken as an Abſciſſa is lower in Power than the

Semiordinate Z M = v. And whenever theſe are equal in Power, it is plain that V M can be no Curve, but muſt coincide with their common Chord or Hypothenuſe, and conſequently divide the Rectangle V P M Z into two congruous right angled Triangles. Therefore every greater Area V M P is more than half its circumſcribing Rectangle, and every leſſer Area V M Z is leſs than half the ſame. THEOREM LI.

If then in the greater Area V M P we make the Abſciſſa V P = v = a, and its correſpondent Semiordinate P M = s = $a^{\frac{1}{r}}$; it is manifeſt, if a be ſuppoſed infinite, that the whole is made up of an Infinity of Semiordinates, each of which is the $\frac{1}{r}$ Power of its reſpective Abſciſſa. Conſequently (putting a for n, and $\frac{1}{r}$ for m in Theo. 48.) we have the greater Area V M P $= \frac{r}{r+1} a^{\frac{r+1}{r}}$; whoſe Ultimator (Theo. 19.) is $a^{\frac{1}{r}} a^{0}$, or (reſtoring s for $a^{\frac{1}{r}}$, and v for a) s v^{0}. That is in all greater Areas, As Unity is to the Semiordinate, ſo is the Ultimator of the Abſciſſa to the Ultimator of the Area of the Curve. THEOREM LII.

So in the leſſer Area V M Z, if we make Z M = v = a^{r}, and V Z = s = a; it is alſo manifeſt, if a be ſuppoſed infinite, that the whole is made up of an Infinity of Lines parallel to Z M, each of which is the r Power of its reſpective Diſtance upon V Z from V. Conſequently, (putting a for n, and r for m in Theo. 50.) we have the leſſer Area V M Z $= \frac{1}{r+1} a^{r+1}$; whoſe Ultimator (Theo. 19.) is $a^{r} a^{0}$, or reſtoring v for a^{r} and s for a) v s^{0}. That is, in all leſſer Areas, as Unity is to the Abſciſſa of the greater Area, ſo is the Ultimator of the Semiordinate of the ſame to the Ultimaror of the leſſer Area. THEOREM LIII.

If therefore in the Ultimator s v^{0} (Theo. 52.) we ſubſtitute $\frac{s s^{0}}{\overline{cc - ss}^{\frac{1}{2}}} = \frac{s s^{0}}{r - y}$ for v^{0} (Theor. 32.) we have s $v^{0} = \frac{s^{2} s^{0}}{\overline{cc - ss}^{\frac{1}{2}}}$

=

$$=\frac{1}{c}s^2s^0+\frac{1}{2c^3}s^4s^0+\frac{3}{8c^5}s^6s^0+\frac{5}{16c^7}s^8s^0+\frac{15}{128c^9}s^{10}s^0$$
$$+\frac{63}{256c^{11}}s^{12}s^0+\frac{231}{1024c^{13}}s^{14}s^0+\frac{429}{2048c^{15}}s^{16}s^0+\frac{6435}{32768c^{17}}s^{18}s^0$$

+ &c. Whoſe Subject is $\frac{1}{3c}s^3+\frac{1}{10c^3}s^5+\frac{3}{56c^5}s^7+\frac{5}{144c^7}s^9$
$$+\frac{35}{1408c^9}s^{11}+\frac{63}{3328c^{11}}s^{13}+\frac{77}{5120c^{13}}s^{15}+\frac{429}{35810c^{15}}s^{17}+\frac{6435}{62592c^{17}}$$
s^{19}+ &c. equal to the greater Area MVP of every Curve univerſally, whoſe Normal is c, and Semiordinate is *s*. Tʜᴇᴏʀᴇᴍ LIV.

In ſuch ſolid Spaces (ſee the laſt Figure) as are generated by the Rotation of a plan Area, as in in the three laſt; whether greater (as M V P about V P or M P) or leſſer (as M V *Z* about M Z or V Z) always put b (as in Theo. 46.) equal to the Diameter of a Circle, whoſe Circumference is q. Then the Area of the Circle, whoſe Radius is s, will be $\frac{q}{b}$ s s, and the Content of the Cylinder upon the ſame Circle, whoſe Height is v, will be $\frac{q}{b}$ s s v. And the Area of the Circle, whoſe Radius v, will be $\frac{q}{b}$ v v; and the Content of the Cylender upon the ſame Circle, whoſe Height is s will be $\frac{q}{b}$ v v s. Tʜᴇᴏʀᴇᴍ LV.

Alſo it appears from the Nature of a Right Cone, that every convex Solid ſo formed muſt be more, and concave Solid leſs than one Third its circumſcribing Cylinder. Tʜᴇᴏʀᴇᴍ LVI.

Firſt then, If the greater Area V M P (ſee the the laſt Figure) be turned about its Abſciſſa V P, as a fixed Axis, I call univerſally the Solid ſo generated, *Convex V*. In this Caſe put V P = v = a, and P M = s = $a^{\frac{1}{r}}$ (Theo. 51, 52.) Whence (Theo. 55.) the Area of the Baſe is $\frac{q}{b}$ $a^{\frac{2}{r}}$, and the Content of the circumſcribing

Cylinder is $\frac{q}{b}a^{\frac{2+r}{r}}$. Now if V P $=$ a be conſidered as divided into an Infinity of Abſciſſa 1, 2, 3, 4, 5, &c. whoſe greateſt Term is V P $=$ a, then the reſpective Semiordinates to thoſe Abſciſſæ will be $1^{\frac{1}{r}}$, $2^{\frac{1}{r}}$, $3^{\frac{1}{r}}$, $4^{\frac{1}{r}}$, $5^{\frac{1}{r}}$, &c. till we come to $a^{\frac{1}{r}}$, whoſe Squares are $1^{\frac{2}{r}}$, $2^{\frac{2}{r}}$, $3^{\frac{2}{r}}$, $4^{\frac{2}{r}}$, $5^{\frac{2}{r}}$, &c. to $a^{\frac{2}{r}}$. And ſince the Areas of Circles are always as the Squares of their Radii; therefore the Sum of the Areas of the Circles of theſe reſpective Radii will be equal to $\frac{q}{b}$ multiplying the Series $1^{\frac{2}{r}}+2^{\frac{2}{r}}+3^{\frac{2}{r}}+4^{\frac{2}{r}}+5^{\frac{2}{r}}+$ &c. to $+\ a^{\frac{2}{r}}$. But the Sum of ſuch a Series of $\frac{2}{r}$ Powers, whoſe laſt Terms is $a^{\frac{2}{r}}$, (by putting a for n, and $\frac{2}{r}$ for m in Theo. 50.) is $\frac{r}{2+r}\ a^{\frac{2+r}{r}}$. Conſequently every *Convex Solid V* is equal to $\frac{r}{2+r}\times\frac{q}{b}a^{\frac{2+r}{r}}$. Whoſe Ultimator (Theo. 19.) is $\frac{q}{b}\ a^{\frac{2}{r}}a^{0} =$ $\frac{q}{b}\ s\,s\,v^{0}$; which gives the following Theorem. As the Diameter b of any Circle is to the Product of the Square s s of the Semiordinate multiplied by the Circumference q of that Circle; ſo is the Ultimator v^{0} of the Abſciſſa, to the Ultimator $\frac{q}{b}\ s\,s\,v^{0}$ of every *Convex Solid V* THEOREM LVII.

The Complement of which to its circumſcribing Cylinder $\frac{q}{b}\ s\,s\,v$ muſt give the Content of the concave Circumambient Solid Z, which is formed by the Rotation of the leſſer Area V M Z about the ſame Axis V P. And the Ultimator of their common Curve Superficies (Theo. 46.) is $\frac{2q}{b}s\,u^{0}$. THEOREM LVIII.

Secondly, If the leſſer Area V M Z (ſee the laſt Figure) be turned about V Z, as a fixed Axis, I call the Solid ſo generated *Concave V.* In this Caſe put V Z $=$ s $=$ a, and Z M $=$ v $=$ a^{r} (Theo. 51, 52.) Whence the Area of the Baſe (Theo. 55.) is $\frac{q}{b}\, a^{2r}$ and the Content of the circumſcribing Cylinder is $\frac{q}{b} a^{2r+1}$. Now if V Z $=$ a be conſidered as divided into an Infinity of Parts 1, 2, 3, 4, 5, &c. $=$ V D $=$ A E, whoſe laſt Term is V Z $=$ A B $=$ a; then the 2 r Powers of their reſpective Perpendiculars DE, ZM will be 1^{2r}, 2^{2r}, 3^{2r}, 4^{2r}, 5^{2r}, to the laſt Term a^{2r}, $= \overline{VP}^{2} = \overline{ZM}^{2}$. And ſince the Areas of Circles are always as the Squares of ~~theſe~~ their Radii; therefore the Sum of the Areas of the Circles of all ~~their~~ theſe reſpective Radii will be equal to $\frac{q}{b}$ multiplying the Series $1^{2r} + 2^{2r} + 3^{2r} + 4^{2r} + 5^{2r}$ &c. $+\, a^{2r}$. But the Sum of ſuch a Series of 2 r Powers, whoſe laſt Term is a^{2r}, (by putting a for n, and 2 r for m Theo. 50.) is $\frac{1}{2r+1}\, a^{2r+1}$. Conſequently, every *Concave Solid* V is equal to $\frac{1}{2r+1} \times \frac{q}{b}\, a^{2r+1}$: Whoſe Ultimator is $\frac{q}{b}\, a^{2r} a^{0} = \frac{q}{b} v v^{0}$. Whence, As the Diameter b of any Circle is to the Product of the Square $\overline{ZM}^{2} = v\,v$ multiplied by the Circumference q of that Circle; ſo is the Ultimator s^{0} of V Z to the Ultimator of every *Concave Solid* V. Theorem LIX.

The Complement of which to its circumſcribing Cylinder $\frac{q}{b}\, v v s$ muſt give the Content of the convex circumambient Solid P which is formed by the Rotation of the greater Area VMP about the ſame Axis V Z. And (putting M Z $=$ d fixed, and M B $=$ v variable, whence E D $=$ d $-$ v) the Ultimator of their common curve Superfices will be $\frac{2qu^{0}}{b} \times \overline{d - v}$, reſtoring v for d in the Subject. Theorem LX.

Thirdly, If the greater Area M V P (ſee the laſt Figure) be turned about its Semiordinate M P, as a fixed Axis, I call the Solid, ſo generated, *Convex* M. In this Caſe put again $MP = s = a$, and $VP = v = a^r$ (Theo. 51. 52): whence (Theo. 55.) the Area of the Baſe is $\frac{q}{b} a^{2r}$, and the Content of the circumſcribing Cylinder is $\frac{q}{b} a^{2r+1}$. Now in the *Concave Solid* V it was juſt now ſhewn that all its Semiordinates D E, Z M (to the ſame Axis $VZ = MP = a = s$) were 1^r, 2^r, 3^r, 4^r, 5^r, &c. to a^r. Therefore every E F, being here taken as a Semiordinate, as the Complements of all thoſe to the greateſt $VP = ZM$, muſt give all ſuch Semiordinates E F equal to $a^r - 1^r$, $a^r - 2^r$, $a^r - 3^r$, $a^r - 4^r$, $a^r - 5^r$, &c. decreaſing *ad infinitum*, till all terminate in the Point $M = a^r - a^r = 0$. Hence the Squares of all theſe Semiordinates (or Radii of their reſpective Circles) are $a^{2r} - 2a^r \times 1^r + 1^{2r}$, $a^{2r} - 2a^r \times 2^r + 2^{2r}$, $a^{2r} - 2a^r \times 3^r + 3^{2r}$, $a^{2r} - 2a^r \times 4^r + 4^{2r}$, &c. And each multiplied by $\frac{q}{b}$ are the Areas of their reſpective Circles. But it is evident, that the Sum of all the firſt Terms of theſe circular Areas is $\frac{q}{b} \times MP \times \overline{VP}^2 = \frac{q}{b} a^{2r+1}$, the ſame with the circumſcribing Cylinder. Conſequently the Sums of all the reſt, under their contrary Signs, muſt be the Complement of the *Convex Solid* M to its circumſcribing Cylinder; which is the Content of the *circumambient Diſcus*, formed at the ſame Time by the Rotation of the concave Area M V Z about the ſame Axis PM. Now the former of theſe Sums is equal to $\frac{q}{b} \times 2a^r$ multiplying the Series $1^r + 2^r + 3^r + 4^r + 5^r$ &c. $+ a^r$, which (Theo. 50.) is $\frac{2qa^r}{b} \times \frac{1}{r+1} a^{r+1}$. And the latter is equal to $\frac{q}{b}$ multiplying the Series $1^{2r} + 2^{2r} + 3^{2r} + 4^{2r} + 5^{2r}$ &c. $+ a^{2r}$, which (Theo. 50.) is $\frac{q}{b} \times \frac{1}{2r+1} a^{2r+1}$. Therefore the Content of the ſaid *Diſcus* muſt equal $\frac{2qa^r}{b} \times \frac{1}{r+1} a^{r+1} - \frac{q}{b} \times \frac{1}{2r+1} a^{2r+1}$. But obſerve here, that the $2a^{r} = 2v$ in the former Part of the former Term, is always as invariable to all its variable Coefficients (*viz.* 1^{r}, 2^{r}, 3^{r}, 4^{r}, 5^{r}, &c.) as the fixed Coefficient 2 itſelf. Therefore (Theo. 42.) the ſaid Content is equal to $\frac{2q\dot{v}}{b} \times \frac{1}{r+1}\,a^{r+1} - \frac{q}{b} \times \frac{1}{2r+1}\,a^{2r+1}$. Whoſe Ultimator (Theo. 19.) is $\frac{2q\dot{v}}{b}\,a^{r}\,a^{0} - \frac{q}{b}\,a^{2r}\,a^{0}$, or (every where reſtoring v for a^{r} and s for a) $\frac{2q\dot{v}}{b}\,v\,s^{0} - \frac{q}{b}\,v^{2}\,s^{0}$ equal to the Ultimator of the Complement of the *Convex Solid* M to its circumſcribing Cylinder. But $\frac{q}{b}\,v^{2}\,s^{0}$ (by the laſt but one) is the Ultimator of the *Concave Solid V*. Therefore the circumambient concave Solid Z, which is formed by the Rotation of the leſſer Area V M Z about the Semiordinate P M, is equal to the Difference, which remains after the Subtraction of the concave Solid V, found as in the laſt, from the Subject of the Ultimator $\frac{2q\dot{v}}{b}\,v\,s^{0}$; always obſerving, that in finding that Subject, the Quantity $\dot{v}$ is as invariable in it ſelf, as q or b, but not v. Tʜᴇᴏʀᴇᴍ LXI.

Hence it follows, that the *convex Solid* M, which is formed by the Rotation of the greater Area V M P about the ſame Semiordinate P M is equal to the Difference, which remains, after the Subtraction of the Subject of the Ultimator $\frac{2q\dot{v}}{b}\,v\,s^{0}$, from the Sum of the *concave Solid* V (Theo. 58.) added to the circumſcribing Cylinder $\frac{q}{b}\,v\,v\,s$. And the Ultimator of the common curve Superfices of theſe two laſt (Theo. 48.) is $\frac{2q}{b}\,v\,\dot{v}^{0}$. Tʜᴇᴏʀᴇᴍ LXII.

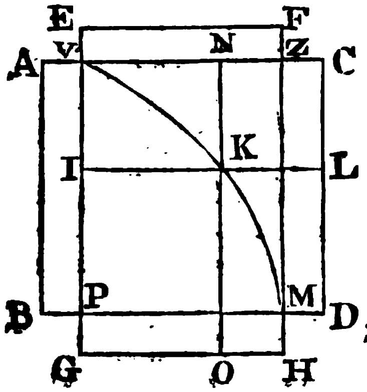
Fourthly, If the leſſer Area V M Z (ſee the laſt Figure) be turned about the Axis M Z, as a fixed Axis, I call the Solid, ſo generated *Concave* M. In this Caſe (as in the firſt) put MZ $= VP = v = a$, and $VZ = PM = s = a^{\frac{1}{r}}$ (Theo. 51. 52.): Whence the Area of the Baſe and Content of the circumſcribing Cylinder muſt be the ſame as in the firſt. Now in the *Convex Solid* V it was ſhewn (Theo. 57.) that all its Semiordinates A E, PM (to the ſame Axis $VP = ZM = a = v$) were $1^{\frac{1}{r}}, 2^{\frac{1}{r}}, 3^{\frac{1}{r}}, 4^{\frac{1}{r}}, 5^{\frac{1}{r}}$, &c. to $a^{\frac{1}{r}}$. Therefore every E B being here taken as a Semiordinate, as the Complements of all thoſe to the greateſt $VZ = PM$, muſt give all ſuch Semiordinates E B equal to $a^{\frac{1}{r}} - 1^{\frac{1}{r}}$, $a^{\frac{1}{r}} - 2^{\frac{1}{r}}$, $a^{\frac{1}{r}} - 3^{\frac{1}{r}}$, $a^{\frac{1}{r}} - 4^{\frac{1}{r}}$, $a^{\frac{1}{r}} - 5^{\frac{1}{r}}$, &c. decreaſing *ad infinitum* till all terminate in the Point $M = a^{\frac{1}{r}} - a^{\frac{1}{r}} = 0$. Hence the Sqnares of all theſe Semiordinates (or Radii of their reſpective Circles) are $a^{\frac{2}{r}} - 2a^{\frac{1}{r}} \times 1^{\frac{1}{r}} + 1^{\frac{2}{r}}$, $a^{\frac{2}{r}} - 2a^{\frac{1}{r}} \times 2^{\frac{1}{r}} + 2^{\frac{2}{r}}$, $a^{\frac{2}{r}} - 2a^{\frac{1}{r}} \times 3^{\frac{1}{r}} + 3^{\frac{2}{r}}$, $a^{\frac{2}{r}} - 2a^{\frac{1}{r}} \times 4^{\frac{1}{r}} + 4^{\frac{2}{r}}$, &c. And each multiplied by $\frac{q}{b}$ are the Areas of the Circles. But it is evident that the Sum of all the firſt Terms of theſe circular Areas is $\frac{q}{b} \times MZ \times VZ^2 = \frac{q}{b} \times a^{\frac{2+r}{r}}$, the ſame with the circumſcribing Cylinder. Conſequently the Sums of all the reſt under their contrary Signs, muſt be the Complement of the *Concave Solid* M to its circumſcribing Cylinder, which is the Content of the circumambient convex Solid, formed at the ſame Time by the Rotation of the greater Area M V P, about the ſame Axis ZM. Now the former of theſe Sums is equal to $\frac{q}{b} \times 2a^{\frac{1}{r}}$ multiplying the Series $1^{\frac{1}{r}} + 2^{\frac{1}{r}} + 3^{\frac{1}{r}} + 4^{\frac{1}{r}} + 5^{\frac{1}{r}}$ &c. to $+ a^{\frac{1}{r}}$, which, (Theo. 50.) is $\frac{2qa^{\frac{1}{r}}}{b} \times \frac{r}{1+r} a^{\frac{1+r}{r}}$, And the latter is equal to $\frac{q}{b}$ multiplying the Series $1^{\frac{2}{r}} + 2^{\frac{2}{r}} + 3^{\frac{2}{r}} + 4^{\frac{2}{r}} + 5^{\frac{2}{r}}$ &c. to $+ a^{\frac{2}{r}}$, which (Theo. 48.) is $\frac{q}{b} \times \frac{r}{2+r} a^{\frac{2+r}{r}}$, Therefore the Content of the ſaid convex circumambient Solid muſt equal $\frac{2qa^{\frac{1}{r}}}{b} \times \frac{r}{1+r} a^{\frac{1+r}{r}} + \frac{q}{b} \times \frac{r}{2+r} a^{\frac{2+r}{r}}$. But again obſerve here, that the $2a^{\frac{1}{r}} = 2s$ in the former Part of the former Term is always as invariable to all its variable Coefficients (*viz.* $1^{\frac{1}{r}}$, $2^{\frac{1}{r}}$, $3^{\frac{1}{r}}$, $4^{\frac{1}{r}}$, $5^{\frac{1}{r}}$, *&c.*) as the fixed Coefficient 2 itſelf. Therefore (Theo. 42.) the ſaid Content is equal to $\frac{2q\dot{s}}{b} \times \frac{r}{1+r} a^{\frac{1+r}{r}} - \frac{q}{b} \times \frac{r}{2+r} a^{\frac{2+r}{r}}$. Whoſe Ultimator (Theo. 19.) is $\frac{2q\dot{s}}{b} a^{\frac{1}{r}} a^{0} - \frac{q}{b} a^{\frac{2}{r}} a^{0} = \frac{2q\dot{s}}{b} s v^{0} - \frac{q}{b} s s v^{0}$ equal to the Ultimator of the Complement of the *Concave Solid* M. to its circumſcribing Cylinder. But $\frac{q}{b} s s v^{0}$ (Theo. 57.) is the Ultimator of the convex Solid V. Therefore the circumambient convex Solid P, which is foamed by the Rotation of the greater Area VMP about the Axis MZ is equal to the Difference, which remains after the Subtraction of the *Convex Solid* V, from the Subject of the Ultimator $\frac{2q\dot{s}}{b} s v^{0}$: always obſerving the $\dot{s}$ fixed here, as the $\dot{v}$ before. THEOREM LXIII.

Hence it follows that the *Concave Solid* M, which is formed by the Rotation of the leſſer Area VMZ, about the ſame Axis MZ, is equal to the Difference, which remains, after the Subtraction of the Subject of the Ultimator $\frac{2q\dot{s}}{b} s v^{0}$, from the Sum of the *Convex Solid* V (Theo. 57.) added to the circumſcribing Cylinder $\frac{q}{b} s s v$. And (putting $VZ = d$ fixed, and $VD = s$ variable, whence $BE = d - s$) the Ultimator of the common curve Superfices of theſe two laſt will be $\frac{2qu^{0}}{b} \times \overline{d-s}$, reſtoring s for d in the Subject. THEOREM LXIV.

Next, to determine the Contents of ſuch convex and concave Solids, as are formed by the Rotation of curve Areas, at the fixed Diſtance l from their reſpective Axes. Theſe when taken wholly, I call *Mixed Solids*, and when the Right Cylinder is taken out of each, the Radius of whoſe Baſe is l, and the Height is the given Axis, I call the Remainders ſimply *Ambients*, Thus,

Firſt, If the greater Area V M P (ſee the annexed Figure) be turned about the Axis A B parallel and equal to its Abſciſſa V P $= v$, at the Diſtance A V $=$ B P $=$ l. Then, proceeding with B M $= \text{l} + \text{s}$ directly as with s in Theo. 57, we have the Ultimator of the mixed convex Solid A equal to $\frac{qv^{\circ}}{b} \times \overline{\text{l}+\text{s}}^{2}$; from whoſe Subject, ſubtracting the Cylinder $\frac{q}{b}\,\text{l}^{2}\,v$, we have the Convex Ambient V M P. Theorem LXV.

And, if from the Cylinder $\frac{qv}{b} \times \overline{\text{l}+\text{s}}^{2}$, we ſubtract the ſaid mix'd Convex Solid A, we have the Content of the concave Ambient V Z M, which is formed at the ſame Time by the Rotation of the leſſer Area V M Z about the ſame Axis A B $= v$. And the Ultimator of their common curve Superficies (Theo. 46.) will be $\frac{2\,q\,u^{\circ}}{b} \times \overline{\text{l}+\text{s}}$. Theorem LXVI.



Secondly, If the leſſer Area V M Z be turned about the Axis E F parallel and equal to the Semiordinate P M $=$ V Z $=$ s, at the Diſtance E V $=$ F Z $=$ l. Then, proceeding with E P $= \text{l} + v$ directly as with v in Theo. 59, we have the Ultimator of the mixed concave Solid E equal to $\frac{qs^{\circ}}{b} \times \overline{\text{l}+v}^{2}$; from whoſe Subject,

ject, ſubtracting the Cylinder $\frac{q}{b}lls$, we have the correſpondent concave Ambient V M Z. Theorem LXVII.

And if from the Cylinder $\frac{qs}{b}\times\overline{l+v}^2$ we ſubtract the ſaid mixed concave Solid E, we have the Content of the convex Ambient V P M; which is formed at the ſame Time by the Rotation of the greater Area V M P about the ſame Axis E F $=s$. And the Ultimator of their common curve Superficies will be $\frac{2qu^{o}}{b}\times\sqrt{\overline{l+\dot{v}}}-v$. Theorem. LXVIII.

Thirdly, If the greater Area M V P be turned about the Axis G H parallel and equal to the Semiordinate P M $=s$, at the Diſtance P G $=$ M H $=l$. Then, to find the Value of the concave Ambient M Z V, formed at the ſame Time by the Rotation of the leſſer Area V M Z, about the ſame Axis G H $=s$, proceed as follows. Put G H $=s=a$, and V P $=v=a^r$: Whence the Radius of the Baſe V G is $l+a^r$, the Area of the Baſe is $\frac{q}{b}l^2+\frac{2q}{b}la^r+\frac{q}{b}a^{2r}$, and the Content of the circumſcribing Cylinder is $\frac{q}{b}l^2a+\frac{2q}{b}la^{r+1}+\frac{q}{b}a^{2r+1}$. Now in the concave Area V M Z, here as before all its Semiordinates N K to Z M are 1^r, 2^r, 3^r, 4^r, 5^r, &c. to Z M $=a^r$. Therefore every K O, being here taken as a Radius as the Complements of all thoſe to the greateſt V G $=l+a^r$, muſt give all ſuch Radii K O equal to $l+a^r-1^r$, $l+a^r-2^r$; $l+a^r-3^r$, $l+a^r-4^r$, &c. decreaſing *ad infinitum* till all terminate in the Point M $=l+a^r-a^r=l=$ M H. Hence the Squares of all theſe Radii are $\overline{l+a^r}^2-\overline{2a^r+2l}\times1^r+1^{2r}$, $\overline{l+a^r}^2-\overline{2a^r+2l}\times2^r+2^{2r}$, $\overline{l+2^r}^2-\overline{2a^r+2l}\times3^r+3^{2r}$, &c. And each multiplied by $\frac{q}{b}$ are the Areas of their reſpective Circles. But it is evident, that the Sum of all the Terms $\frac{q}{b}\times\overline{l+a^r}^2$ of theſe circular Areas is $\frac{q}{b} \times GH \times \overline{VG}^2 = \frac{b}{b} l^2 a + \frac{2q}{b} l a^{r+1} + \frac{q}{b} a^{2r+1}$, the circumſcribing Cylinder. Conſequently, the Sums of the two other Terms under their contrary Signs, muſt be the Complement of the mixed convex Solid M here, to its circumſcribing Cylinder. Whence, proceeding directly as in Theo. 59. we have $\frac{2q}{b} \times \overline{l + \dot{v}} \times v\, s^0 - \frac{q}{b}\, v\, v\, s^0$ for the Ultimator of the concave Ambient M Z V formed by the Rotation of the leſſer Area V M Z about the Axis G H. Therefore the Content of every ſuch Ambient, is equal to the Remainder, which is left, after the Subtraction of the concave Solid V, (Theo. 59.) from the Subject of the Ultimator $\frac{2q}{b} \times \overline{l + \dot{v}} \times v\, s^0$. THEOREM LXIX.

And if from the Sum of the laſt Cylinder $\frac{qa}{b} \times \overline{l + a^r}^2 = \frac{qs}{b} \times \overline{l + v}^2$ added to the ſaid concave Solid V, we take the Subject of the Ultimetor $\frac{2q}{b} \times \overline{l + \dot{v}} \times v s^0$; the Remainder will be the Content of the mixed Solid H, which is formed by the Rotation of the mixed greater Area G V K M H, about the ſame Axis G H $= s$. From which laſt again taking the Cylinder $\frac{q}{b}\, l\, l\, s$, we have the correſpondent convex Ambient M V P. And the Ultimator of their common curve Superficies (Theo. 48.) will be $\frac{2qu^0}{b} \times \overline{l + v}$. THEOREM LXX.

Fourthly, By proceeding again, as in Theo. 67. compared with Theo. 63. we ſhall the ſame Way find the Content of the convex Ambient M P V, which is formed by the Rotation of the greater Area V M P, about the Axis C D $= v$, to be equal to the Remainder, which is left, after the Subtraction of the convex Solid V (Theo. 57.) from the Subject of the Ultimator $\frac{2q}{b} \times \overline{l + \dot{s}} \times s v^0$. THEOREM LXXI.

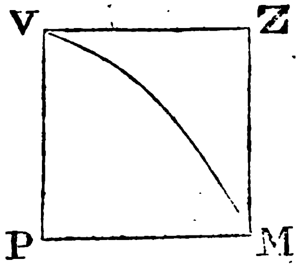
And if from the Sum of the Cylinder $\frac{qv}{b} \times \overline{1+s}^2$ added to the ſaid convex Solid V, (Theo. 57.) we take the Subject of the Ultimator $\frac{2q}{b} \times \overline{1+\dot{s}} \times s\, v^o$; the Remainder will be the Content of the mixed concave Solid D, which is formed by the Rotation of the mixed leſſer Area D M K V C about the ſame Axis C D $= v$. From which laſt again, taking the Cylinder $\frac{q}{b}$ 11 v, we have the correſpondent concave Ambient M V Z. And the Ultimator of their common curve Superficies will be $\frac{2qu^o}{b} \times \overline{1+\dot{s}-s}$.

Theorem LXXII.

Proceed we now to give ſome Examples to the foregoing Theorems.

Example 1. To determine the Ratio of the greater Area P V M, and leſſer Area M V Z of the common * Parabola, to their circumſcribing Rectangle P V Z M.

The Equation of this Curve, is $2\,p\,v =$ [illegible] s. Whence $v = \frac{ss}{2p}$. Therefore for the Ultimator of the greater Area, we have $v^o = \frac{1}{p}\, s\, s^o$, and (Theo. 52.) $s\,v^o = \frac{1}{p}\, s^2\, s^o$. Whoſe Subject (Theo. 50.) is $\frac{1}{3p}\, s^3$. But $s^2 = 2pv$. Therefore $\frac{1}{3p}\, s^3 = \frac{2}{3}\, s\, v$ the greater Area P V M; and conſequently $\frac{1}{3}\, s\, v$ equals the leſſer Area M V Z. That is, the former equals two Thirds, and the latter one Third of the circumſcribing Rectangle P V Z M.



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* *See* Parabola *in the Appendix.*

Example 2. In the ſame Parabola, to determine the Ratio between the circumſcribing Cylinder and the convex Solid V; which is formed by the Rotation of the greater Area V M P about the Axis $VP = v$.

From the Equation of the Curve, we have $2pv = ss$. Whence (Theo. 57.) $b : q \times 2pv = v^0 : \frac{2qp}{b} v v^0$ the Ultimator of the Solid V. Whoſe Subject is $\frac{qp}{b} vv$. (Theo. 20.) But $pv = \frac{1}{2} ss$, therefore the Content required is $\frac{q}{2b} ssv$, or (Theo. 55.) the one Half of its circumſcribing Cylinder. Conſequently, the circumambient Solid formed at the ſame Time by the leſſer Area V M Z (Theo. 58.) muſt be the other Half.

Example 3. In the ſame Parabola, to determine the Content of the Solid V; which is formed by the Revolution of the leſſer Area V M Z, about the Axis $VZ = s$.

The Ultimator of this Solid, from $2pv = s^2$, is (Theo. 59.) $\frac{q}{4bpp} s^4 s^0$. Whoſe Subject (Theo. 20.) is $\frac{qs^5}{20bpp}$, or (ſubſtituting $4p^2 s^2$ for s^4) the ſame Subject is $\frac{q}{5b} v^2 s$. That is (Theo. 55.) one fifth Part of its circumſcribing Cylinder for the Content required. Conſequently, the Solid formed at the ſame Time by greater Area V M P (Theo. 60.) is $\frac{4q}{5b} v^2 s$, or four Fifths of the ſame Cylinder.

Example 4. In the ſame Parabola, to determine the Content of the convex Solid M; which is formed by the Rotation of the greater Area V M P, about the Axis $PM = s$.

Here the Sum of the circumſcribing Cylinder (Theo. 55.) $\frac{q}{b} vvs$ and concave Solid V equal to $\frac{q}{5b} vvs$ (found in the laſt

Example) is $\frac{6q}{5b}vvs$. And becauſe $v=\frac{ss}{2p}$; therefore the Ultimator $\frac{2q\dot{v}}{b}vs^{\circ}$ (Theo. 62,) is $\frac{q\dot{v}}{bp}s^{2}s^{\circ}$, whoſe Subject with $\dot{v}$ fixed is $\frac{qv}{3bp}s^{3}$, or (ſubſtituting $2pv$ for ss) $\frac{2q}{3b}vvs$. Which, (Theo. 60.) ſubtracted from the Sum $\frac{6q}{5b}vvs$, leaves $\frac{8q}{15b}vvs$; or eight Fifteenths of the circumſcribing Cylinder, for the Content required. Conſequently, the Solid formed at the ſame Time by the leſſer Area V M Z is $\frac{7q}{15b}vvs$, or ſeven Fifteenths of the circumſcribing Cylinder.

Example 5. In the ſame Parabola, to determine the Content of the Concave Solid M; which is formed by the Rotation of the leſſer Area V M Z about the Axis $MZ=v$.

Here the Sum of the circumſcribing Cylinder $\frac{q}{b}ssv$ and the convex Solid Vequal to $\frac{q}{2b}ssv$ (found in the ſecond Example) is $\frac{3q}{2b}ssv$. And becauſe in the Parabola $v^{\circ}=\frac{ss^{\circ}}{p}$; therefore the Ultimator $\frac{2q\dot{s}}{b}sv^{\circ}$ (Theo. 64.) is $\frac{2q\dot{s}}{bp}s^{2}s^{\circ}$ whoſe Subject with $\dot{s}$ fixed is $\frac{2qs^{4}}{3bp}$, or (ſubſtituting $2pv$ for ss) $\frac{4q}{3b}s^{2}v$. Which, (Theo. 64.) ſubtracted from the Sum $\frac{3q}{3b}s^{2}v$, leaves $\frac{q}{6b}s^{2}v$; or one ſixth of the circumſcribing Cylinder, for the Content required. Conſequently, the Solid formed at the ſame Time by the greater Area V M P is $\frac{5q}{6b}s^{2}v$, or five Sixths of the circumſcribing Cylinder.

Example 6. Suppoſing the foregoing Figure to be the Portion of a common * Ellipſis, whoſe Equation is $\frac{2cc}{d}v - \frac{c\,c}{d\,d}v\,v = ss$; let it be required to determine the Ratio of the greater Area V M P and leſſer Area V M Z, to their circumſcribing Rectangle P V Z M.

Here we have $v = d - \frac{d}{c} \times \overline{cc - ss}^{\frac{1}{2}}$, Whence (Theo. 17, 21.) $\overset{\circ}{v} = \frac{d}{c} \times \frac{s\overset{\circ}{s}}{\overline{cc - ss}^{\frac{1}{2}}}$, and $s\overset{\circ}{v}$ (Theo. 50.) the Ultimator of the Area required becomes $\frac{d}{c} \times \frac{s^2\overset{\circ}{s}}{\overline{cc - ss}^{\frac{1}{2}}}$ equal to $\frac{d}{c}$ multiplying the Series $\frac{1}{c}s^2\overset{\circ}{s} + \frac{1}{2c^3}s^4\overset{\circ}{s} + \frac{3}{8c^5}s^6\overset{\circ}{s} +$, &c. Whoſe Subject is $\frac{d}{c}$ multiplying the Series $\frac{1}{3c}s^3 + \frac{1}{10c^3}s^5 + \frac{3}{56c^5}s^7 +$, &c. (See Theo. 54.) the greater Area required.

Therefore when this elliptic Area is the Quadrant of an Ellipſis, in which Caſe s and c the Semiconjugate are the ſame; its Value muſt be equal to d c multiplied by the Series $\frac{1}{3} + \frac{1}{10} + \frac{3}{56} + \frac{5}{144} + \frac{35}{1408} + \frac{63}{3328} + \frac{77}{5120} + \frac{429}{35816} + \frac{6435}{622592} +$ &c. Conſequently, if we put d for the Radius of a Circle equal to the Semitranſverſe of the Ellipſis here, and ſubſtitute that the ſame Way for c and s, Theo, 54. we ſhall have the Area of that circular Quadrant equal to d d multiplied hy the ſame Series $\frac{1}{3} + \frac{1}{10} + \frac{3}{56} +$ &c. Whence it follows, that the Area of every Circle, whoſe Radius is d, is to the Area of every Ellipſis, whoſe Semitranſverſe is the ſame; as the tranſverſe Diameter of that Ellipſis is to its conjugate Diameter.

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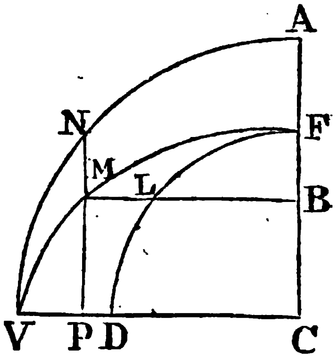
* *See* Ellipſis *in the Appendix.*

Example 7. Suppoſe the annexed Figure A N V C to be the Quadrant of a Circle, whoſe Radius V C = d is the ſame with the Semi-tranſverſe V C of the inſcribed Elliptic Quadrant F M V C, whoſe Semi-conjugate is F C = c; and put V P = v, P M = s, and P N = z. From whence let it be required to find the Proportion between every circular Area V N P, and its correſpondent elliptical Area V M P upon the ſame Abſciſſa V P.

In the Circle we have $2dv - vv = zz$. Whence $z = \overline{2dv - vv}^{\frac{1}{2}}$, or (puttiug b = 2 d) $z = \mathrm{b}\,\overline{v - vv}^{\frac{1}{2}}$, and (Theo. 52.) the Ultimator of the circumſcribing circular Area is $\overline{\mathrm{b}v - vv}^{\frac{1}{2}}\,\dot{v}$

$$\dot{v} = \mathrm{b}^{\frac{1}{2}}\, v^{\frac{1}{2}}\, \dot{v} - \frac{v^{\frac{3}{2}}\,\dot{v}}{2\mathrm{b}^{\frac{1}{2}}} - \frac{v^{\frac{5}{2}}\,\dot{v}}{8\mathrm{b}^{\frac{3}{2}}} - \frac{v^{\frac{7}{2}}\,\dot{v}}{16\mathrm{b}^{\frac{5}{2}}}$$

$$- \frac{5v^{\frac{9}{2}}\,\dot{v}}{128\mathrm{b}^{\frac{7}{2}}} - \frac{7v^{\frac{11}{2}}\,\dot{v}}{256\,\mathrm{b}^{\frac{9}{2}}} - \frac{21v^{\frac{13}{2}}\,\dot{v}}{1024\mathrm{b}^{\frac{11}{2}}} - \&c.$$



Whoſe Subject (Theo. 20.) $\frac{2}{3}\mathrm{b}^{\frac{1}{2}} v^{\frac{3}{2}}$ leſs $\frac{v^{\frac{3}{2}}}{\mathrm{b}^{\frac{1}{2}}}$ multiplying the Series $\frac{v}{5} + \frac{v^2}{28\mathrm{b}} + \frac{v^3}{72\mathrm{b}^2} + \frac{5v^4}{704\mathrm{b}^3} + \frac{7v^5}{1664\mathrm{b}^4} + \frac{7v^6}{2560\mathrm{b}^5} + \&c.$ equals the circular Area V N P.

In the Ellipſis (as before) we have $\mathrm{s} = \mathrm{c} \times \overline{\frac{2}{\mathrm{d}}v - \frac{vv}{\mathrm{dd}}}^{\frac{1}{2}}$, or (putting $\mathrm{b} = 2\mathrm{d}$) $\mathrm{s} = \frac{2\mathrm{c}}{\mathrm{b}} \times \overline{\mathrm{b}v - vv}^{\frac{1}{2}}$. Whence it is manifeſt, that the Correſpondent inſcribed elliptical Area V M P, muſt be the ſame with the former Area multiplied by $\frac{2\mathrm{c}}{\mathrm{b}}$ or $\frac{\mathrm{c}}{\mathrm{d}}$. Conſequent-every elliptical Segment cut perpendicularly to the tranſverſe Diameter is to its correſpondent Segment cut the ſame Way from its circumſcribing Circle, as the Conjugate Diameter 2 c is to the Tranſverſe Diameter 2 d.

Example 8. In the ſame Figure, ſuppoſe F L D C to be the Quadrant of a Circle, whoſe Radius F C $=$ c is the ſame with the Semi-conjugate of the circumſcribing Elliptical Quadrant FMVC, whoſe Semi-tranſverſe is V C $=$ d; and put F B $=$ F C $-$ M P $=$ x, M B $=$ V C $-$ V P $=$ y, and L B $=$ z. From hence it is required to find the Proportion between every circular Area F L B and its correſpondent elliptical one F M B upon the ſame Abſciſſa F B.

Here putting $b = 2c$, in the Circle we have $bx - x^2 = z^2$. Whence as before, the inſcribed circular Area F L B muſt be $\frac{2}{3} b^{\frac{1}{2}} x^{\frac{3}{2}}$ leſs $\frac{x^{\frac{5}{2}}}{b^{\frac{1}{2}}}$ multiplying the Series $\frac{x}{5} + \frac{x^2}{28b} + \frac{x^3}{72b^2} + \frac{5x^4}{704b^3}$ $+$ &c.

And in the Ellipſis, becauſe $FB = x = c - s$, and $MB = y = d - v$; the foregoing Equation of the Curve (by ſubſtituting $c - x$ for s, and $d - y$ for v) becomes $\frac{2dd}{c} x - \frac{dd}{cc} xx = yy$. Whence $y = d \times \overline{\frac{2}{c} x - \frac{xx}{cc}}^{\frac{1}{2}}$, or (putting $b = 2c$) $y = \frac{2d}{b} \times \overline{bx - xx}^{\frac{1}{2}}$. Whence again it is manifeſt, that the correſpondent circumſcribing elliptical Area F M B muſt be the ſame with the former Area multiplied by $\frac{2d}{b}$ or $\frac{d}{c}$. Conſequently, every elliptical Segment cut perpendicularly to the conjugate Diameter is to its correſpondent Segment cut the ſame Way from its ininſcribed Circle, as the Tranſverſe Diameter 2 d is to the Conjugate Diameter 2 c.

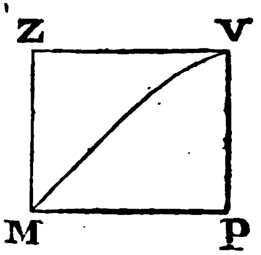
Example 9. Let it be required to find the Content of the convex Solid V, (ſee the laſt Figure) which is formed by the Rotation of the greater elliptical Area V M P about its Semi-tranſverſe V C.

The Equation of the Curve may be $\frac{2cc}{d}v - \frac{cc}{dd}vv = ss$. Whence (Theo. 57.) $b : qcc \times \sqrt{\frac{2v}{d} - \frac{vv}{dd}} = v^{\circ} : \frac{2qc^2vv^{\circ}}{bd} - \frac{qc^2v^2v^{\circ}}{bdd}$ the Ultimator of the Content required. Whoſe Subject (Theo. 20.) is $\frac{qc^2v^2}{bd} - \frac{qc^2v^3}{3bdd}$ or $\frac{qc^2v^2}{3bd^2} \times \overline{3d - v}$. But the circumſcribing Cylinder of this Content (Theo. 56.) is $\frac{q}{b}ssv$, or (ſubſtituting $\frac{2ccv}{d} - \frac{ccvv}{dd}$ for ss;) $\frac{2qc^2dv^2 - qc^2v^3}{dd} = \frac{qc^2v^2}{3bd^2} \times \overline{6d - 3v}$. Conſequently, as $6d - 3v$ is to $3d - v$, ſo is the circumſcribing Cylinder to the Content of every ſuch convex elliptic Solid. Hence,

If this elliptical Area be the Quadrant V M F C, in which Caſe v is the ſame with d; ſo that $6d - 3v = 3d$, and $3d - v = 2d$; it is plain that the cylindrical Content muſt be to the inclnded elliptical one, as three to two.

Example 10. Let it be required to find the Content of the concave Solid M, which is formed by the Rotation of the common leſſer Hyperbolic Area V M Z about the Axis Z M parallel and equal to the Abſciſſa $VP = v$.

Here the Equation of the Curve * is $\frac{2ccv}{d} + \frac{ccvv}{dd} = ss = \overline{MP}^2$.



Whence proceeding directly as in the laſt, the convex Solid V will be found to equal $\frac{qc^2v^2}{3bd^2} \times \overline{3d + v}$, and the circumſcribing Cylinder equal to $\frac{qc^2v^2}{3bd^2} \times \overline{6d + 3v}$. Therefore the Sum of theſe two muſt equal $\frac{qc^2v^2}{3bd^2} \times \overline{9d + 4v}$. Then the Ultimator

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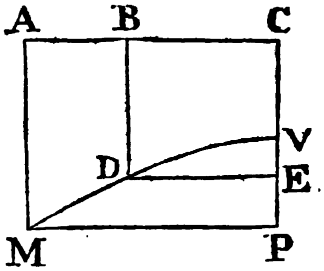
* *See* Hyperbola *in the* Appendix.

timator $\frac{2q\dot{s}c}{b}s\,v^{0}$ (Theo. 64.) will be $\frac{2q\dot{s}c}{bd}\times\overline{2dv+vv}^{\frac{1}{2}}v^{0}$, or (putting $b=2d$) $\frac{2q\dot{s}c}{bd}\times\overline{bv+vv}^{\frac{1}{2}}\,v^{0}=\frac{2q\dot{s}c}{bd}$ multiplying the Series $b^{\frac{1}{2}}v^{\frac{1}{2}}v^{0}+\frac{v^{\frac{3}{2}}v^{0}}{2b^{\frac{1}{2}}}-\frac{v^{\frac{5}{2}}v^{0}}{8b^{\frac{3}{2}}}+\frac{v^{\frac{7}{2}}v^{0}}{16b^{\frac{5}{2}}}-\frac{5v^{\frac{9}{2}}v^{0}}{128b^{\frac{7}{2}}}+\frac{7v^{\frac{11}{2}}v^{0}}{256b^{\frac{9}{2}}}-\frac{21v^{\frac{13}{2}}v^{0}}{1024b^{\frac{11}{2}}}+$ &c. Whoſe Subject is $\frac{2qsc}{bd}$ multiplying the Series $\frac{2}{3}b^{\frac{1}{2}}v^{\frac{3}{2}}+\frac{v^{\frac{5}{2}}}{56^{\frac{1}{2}}}-\frac{v^{\frac{7}{2}}}{28b^{\frac{3}{2}}}+\frac{v^{\frac{9}{2}}}{72b^{\frac{5}{2}}}-\frac{5v^{\frac{11}{2}}}{704b^{\frac{7}{2}}}+\frac{7v^{\frac{13}{2}}}{1664b^{\frac{9}{2}}}-\frac{7v^{\frac{15}{2}}}{2560b^{\frac{11}{2}}}+$ &c. But $s=\frac{c}{d}\times\overline{bv+vv}^{\frac{1}{2}}$. Therefore $\frac{2q\dot{s}c}{bd}$ is equal to $\frac{2qcc}{bdd}\times\overline{bv+vv}^{\frac{1}{2}}=\frac{2qcc}{bdd}$ multiplying the Series $b^{\frac{1}{2}}v^{\frac{1}{2}}+\frac{v^{\frac{3}{2}}}{2b^{\frac{1}{2}}}-\frac{v^{\frac{5}{2}}}{8b^{\frac{3}{2}}}+\frac{v^{\frac{7}{2}}}{16b^{\frac{5}{2}}}-\frac{5v^{\frac{9}{2}}}{128b^{\frac{7}{2}}}+\frac{7v^{\frac{11}{2}}}{256b^{\frac{9}{2}}}-\frac{21v^{\frac{13}{2}}}{1024b^{\frac{11}{2}}}$, $+$ &c. Which (multiplied again into the laſt foregoing Series, inſtead of $\frac{2qsc}{bd}$, and reſtoring 2 d for b) gives the ſame Subject equal to $\frac{qc^{2}v^{2}}{3bd^{2}}$ multiplying the Series $8d+\frac{16}{5}v-\frac{2v^{2}}{35d}+\frac{2v^{3}}{105d^{2}}-\frac{8v^{4}}{1155d^{4}}+\frac{48v^{5}}{18013d^{4}}-$ &c. And this laſt ſubtracted from the Sum $\frac{qc^{2}v^{2}}{3bdd}\times\overline{9d+4v}$ leaves $\frac{qc^{2}v^{2}}{3bd^{2}}$ multiplying the Series $d+\frac{4v}{5}+\frac{2v^{2}}{35d}-\frac{2v^{3}}{105d^{2}}+\frac{8v^{4}}{1155d^{3}}-\frac{48v^{5}}{18013d^{4}}+$ &c.

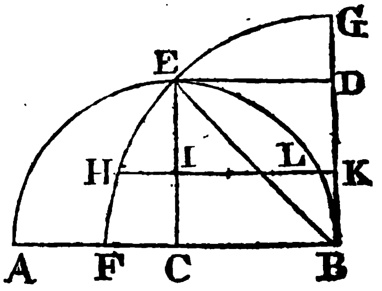
Example 11. Suppoſe CV $=$ d to be the Half tranſverſe Diameter of a common Hyperbola, BC $=$ DE $=$ c its Half conjugate Diameter, and VP $=v$ any Abſciſſa of the ſame with MP $=$ s it Semiordinate: It is required to find the Content of the Solid, which is formed by the Revolution of the Area BDVC about BC $=$ c, as an Axis.

Here the Equation, as before, is $\frac{cc}{dd}\times\overline{2dv+vv}=ss$. Whence $d+v=\frac{d}{c}\times\overline{cc+ss}^{\frac{1}{2}}$. But (Theo. 67.) the Ultimator of the Solid, which is formed at the ſame Time by the Revolution of the Area AMDVC about AC, is $\frac{qs^{0}}{b}\times\overline{d+v}$, becauſe the l there is the ſame with the d here. Therefore, ſubſtituting $\frac{dd}{cc}\times\overline{cc+ss}$ for $\overline{d+v}$, we have $\frac{qdds^{0}}{b}+\frac{qd^{2}s^{2}s^{0}}{bcc}$ for the Ultimator of AMDVC. Whoſe Subject is $\frac{qdds}{b}+\frac{qdds^{3}}{3bcc}$, or (making $b=2d$) $\frac{1}{2}qds+\frac{qds^{3}}{6cc}$. But in the Content required s is the ſame with $DE=BE=c$. The Content therefore of the Solid thus formed by the Rotation of BDVC is $\frac{1}{2}qdc+\frac{1}{6}qdc$ or $\frac{2}{3}qdc$. Which is two Thirds always of the circumſcribing Cylinder.



Example 12. Let it be required to find the Content of the Solid, which is formed by the Rotation of the Area of the annexed Semicircle AEB about the Axis DB, a Tangent perpendicular to its Diameter $AB=2c=b$, at the Point B.

Here (Theo. 55.) the circumſcribing Cylinder muſt be qbc or $2qcc$ or $\frac{1}{2}qbb$. And ſince in determining the ſame Solid, the Solid which is formed by the Rotation of the leſſer Area EBD about DB muſt be added by Theo. 68. and again ſubtracted by the Form of the Solid. Therefore the Ultimator of the Complement of the Solid to its circumſcribing Cylinder (Theo. 70.) muſt be $\frac{2q}{b}\times\overline{c+v}\times vs^{0}$. But in the Circle $bv-vv=ss$. Therefore, beginning the Abſciſſa v from A, the



the Ultimator $2vs^{\circ}$ equals $\frac{bv-vv}{\overline{bv-vv}^{\frac{1}{2}}}v^{\circ}=\frac{bv^{\circ}-2vv^{\circ}}{v^{\frac{1}{2}}}\times\frac{v}{\overline{b-v}^{\frac{1}{2}}}$. But $\frac{v}{\overline{b-v}^{\frac{1}{2}}}$ is equal to $\frac{1}{b^{\frac{1}{2}}}$ multiplying the Series $v+\frac{v^{2}}{2b}+\frac{3v^{3}}{8b^{2}}+\frac{5v^{4}}{16b^{3}}+\frac{35v^{5}}{12\ b^{4}}+\frac{63v^{6}}{250b^{5}}+\frac{231v^{7}}{1024b^{6}}+$ &c. Therefore multiplying this by $\frac{q}{b}\times\overline{\tfrac{1}{2}b+v}\times\frac{b-2v}{v^{\frac{1}{2}}}v^{\circ}$, we have the Ultimator of the Complement equal to $\frac{q}{b^{\frac{3}{2}}}\times\overline{\tfrac{1}{2}b+v}$ multiplying the Series $bv^{\frac{1}{2}}v^{\circ}$ $-\frac{3v^{\frac{3}{2}}v^{\circ}}{2}-\frac{5v^{\frac{5}{2}}v^{\circ}}{8b}-\frac{7v^{\frac{7}{2}}v^{\circ}}{16b^{2}}-\frac{45v^{\frac{9}{2}}v^{\circ}}{128b^{3}}-\frac{77v^{\frac{11}{2}}v^{\circ}}{256b^{4}}-\frac{27v^{\frac{13}{2}}v^{\circ}}{1024b^{5}}$ $-\frac{495v^{\frac{15}{2}}v^{\circ}}{2048b^{6}}-$ &c. Whoſe Subject is $qb^{\frac{1}{2}}v^{\frac{3}{2}}$ multiplying the Series $\frac{1}{3}+\frac{11v}{30b}-\frac{193v^{2}}{280b^{2}}-\frac{229v^{3}}{1008b^{3}}-\frac{1637v^{4}}{11264b^{4}}-\frac{3187v^{5}}{36608b^{5}}-\frac{12789v^{6}}{199680b^{6}}-\frac{25989v^{7}}{522240b^{7}}-$ &c. Conſequently, becauſe in the Solid required, we have $AC=v=c=\frac{1}{2}b$, its Complement will be equal to $\frac{qbc}{2^{\frac{1}{2}}}$ multiplied by the Series. $\frac{1}{3}+\frac{11}{60}-\frac{193}{1120}-\frac{229}{8064}-\frac{1637}{180224}-\frac{3187}{1171456}-\frac{12789}{12779520}-\frac{25989}{66846720}-$ &c. Therefore, ſubtracting this from the circumſcribing Cylinder qbc, we we have $\frac{qbc}{2^{\frac{1}{2}}}$ multiplied by the Series $\frac{60\times2^{\frac{1}{2}}-31}{60}+\frac{193}{1120}+\frac{229}{8064}+\frac{1637}{180224}+\frac{3187}{1171456}+\frac{12789}{12779520}+\frac{25989}{66846720}+$ &c. for the Content required.

Example 13. In the foregoing Figure, ſuppoſe the Arch FE to be deſcribed with the Radius BE upon the Center B: Let it be required to find the Content of the Solid, which is formed by the Rotation of the Semi-lunula A E F about the Axis AF.

Here to find the Content of the Sem-ſphere, which is formed by the Rotation of the Quadrant A E C about the Radius A C, beginning the Abſciſſa from A to C, we have $2cv - vv = ss$. And the Ultimator of the Hemiſphere (Theo. 57.) is $\frac{q}{b} ss\overset{\circ}{v} = \frac{q}{b} \times \overline{2cv\overset{\circ}{v} - v^2 \overset{\circ}{v}}$, whoſe Subject is $\frac{q}{b} \times \overline{cv^2 - \frac{1}{3}v^3}$. But in the Hemiſphere $v = c$. Therefore the Content of that is $\frac{q}{3b} \times \overline{3c^3 - c^3} = \frac{2q}{3b} ccc = \frac{1}{3} qcc$, which is juſt two Thirds of its circumſcribing Cylinder $\frac{q}{b} ccc$.

Next to find the Content of the Solid, which is formed by the Rotation of the greater Area E F C about the Axis F C. Put indefinitely $FC = v$, and $EC = s$. Then juſt as before, we have the Content equal to $\frac{q}{b} \times \overline{cv^2 - \frac{1}{3}v^3}$, or $\frac{qcv^2}{2c} - \frac{qv^3}{6c}$. But v here is equal to $FB - BC = FC = \overline{2^{\frac{1}{2}} - 1} \times c$. Therefore $\frac{qcv^2}{2c} - \frac{qv^3}{6c}$ or $\frac{3qcv^2 - qv^3}{6c} = \frac{qcc}{6} \times \overline{16 - 11 \times 2^{\frac{1}{2}}}$ is the Content of that Solid; which, ſubducted from the foregoing Hemiſphere or $\frac{2}{6} qcc$ leaves $\frac{qcc}{6} \times \overline{11 \times 2^{\frac{1}{2}} - 14}$ for the Content formed by the Semi-lunula, as required.

Example 14. Suppoſing all Things ſtill as in the ſame Figure; let it be required to find the Content of the Ambient Solid, which is formed by the Rotation of the greater Area FEC about the Axis D B, at the Diſtance $CB = ED = c$.

Here it is plain, that $FB = EB = GB = \overline{2cc}^{\frac{1}{2}}$. Whence, by the Anſwer of the laſt Queſtion, the Solid formed by the Rotation of the Quadrant BFEG about $GB = \overline{2cc}^{\frac{1}{2}}$ muſt be the Hemiſphere $\frac{2q}{3b} \times \overline{2cc}^{\frac{3}{2}}$. Then, putting $v =$ any Abſciſſa from G towards B whoſe Semiordinate is s, the Ultimator of the Solid formed at the ſame Time by the Rotation of that greater Area about GB (Theo. 57.) is $\frac{q}{b}\, ss v^{\circ}$. But the Radius of this Circle being $\overline{2cc}^{\frac{1}{2}}$, conſequently from the known Property of the Circle $\overline{2GB - GD} \times GD = \overline{ED}^2$ we have $2v \times \overline{2cc}^{\frac{1}{2}} - vv = ss$. Whence that Ultimator equals $\frac{q}{b} \times \overline{2 \times \overline{2cc}^{\frac{1}{2}}\, v v^{\circ} - v^2 v^{\circ}}$. Whoſe Subject is $\frac{q}{b} \times \overline{\overline{2cc}^{\frac{1}{2}} v^2 - \frac{1}{3} v^3}$. Therefore, if we put $s = ED = c$; whence $GD = v = \overline{2cc}^{\frac{1}{2}} - c$, the ſaid Subject will become $\frac{q}{3b} \times \overline{4c^2 \times \overline{2cc}^{\frac{1}{2}} - 5c^3}$ equal to the Content of the Solid, which is formed by the Rotation of the Area DEG about GD. To which, if we add the included Cylinder $\frac{q}{b} c^3$ formed by the Rotation of the Square BCED about DB, we have $\frac{2q}{3b} \overline{\times 2cc \times \overline{2cc}^{\frac{1}{2}} - c^3} = \frac{2q}{3b} \times \sqrt{\overline{2cc}^{\frac{3}{2}} - c^3}$. Which Sum being ſubtracted from the Hemiſphere $\frac{2q}{3b} \times \overline{2cc}^{\frac{3}{2}}$ leaves $\frac{2q}{3b}\, ccc$, the very ſame with the Hemiſphere in the laſt Queſtion, for the Content required in this.

Otherwiſe this laſt Queſtion may be anſwered, as follows. Draw any Semiordinate $HI = s$ parallel to FC, whoſe Abſciſſa is $EI = DK = v$. Whence, becauſe $GD = \overline{2cc}^{\frac{1}{2}} - c$ we have $GK = \overline{2cc}^{\frac{1}{2}} - c + v$, $HK = c + s$, and $2GB - GK = \overline{2cc}^{\frac{1}{2}} + c - v$. Therefore from the Property of the Circle $\overline{2GB - GK} \times GK = \overline{HK}^{2}$ we have $cc + 2cv - vv = \overline{c+s}^{2}$. But (Theo. 65.) the Ultimator of the Solid formed by the Rotation of the greater Area IHC about the Axis DK, at the Diſtance $IK = c = 1$, is $\frac{q}{b} \times \overline{c+s}^{2}\, v^{\circ}$; which is here $\frac{q}{b} \times \overline{ccv^{\circ} + 2cvv^{\circ} - v^{2}v^{\circ}}$. Whoſe Subject is $\frac{q}{b} \times \overline{c^{2}v + cv^{2} - \frac{1}{3}v^{3}}$. If therefore the $EI = v$ here, be the ſame with $EC = c$, as is ſuppoſed in the Queſtion; then the ſaid Subject muſt equal $\frac{q}{3b} \times 5ccc$. Conſequently, when the included Cylinder $\frac{q}{b}ccc$, as before is ſubtracted, we have $\frac{2q}{3b}ccc$ for the Ambient Solid required, the ſame, as before.

Example 15. Again in the laſt Figure, ſuppoſe $GK = v$ to be any verſed Sine or Abſciſſa, whoſe Sine or Semiordinate is $HK = s$, in the Circle whoſe Radius is $FB = GB = c = \frac{1}{2}b$; and let EI be a Part of any Chord parallel to GK, and HI a Part of HK, whoſe Complement to HK is $IK = ED = l$. It is required to find the Content of the convex Solid, which is formed by the Rotation of the greater Area IEH about the Line EI, as an Axis.

Here $HI = HK - IK = s - l$; or becauſe from the Nature of the Circle $bv - v^{2} = ss$, therefore $HI = \overline{bv - vv}^{\frac{1}{2}} - l$. And for the ſame Reaſon, becauſe $b \times GD - \overline{GD}^{2} = \overline{ED}^{2} = ll$, therefore $GD = \frac{1}{2}b - \overline{\frac{1}{4}bb - ll}^{\frac{1}{2}}$; and conſequently $DK = v$

$-\frac{1}{2}b+\overline{\frac{1}{4}bb-ll}^{\frac{1}{2}}=EI$. Whoſe Ultimator (Theo. 17.) is v^{o}, the ſame with that of the Abſciſſa GK. But (Theo. 55.) As b is to $q\times\overline{HI}^{2}$ or to $q\times\overline{\overline{bv-vv}^{\frac{1}{2}}=l}^{\,2}$, ſo is v^{o} the Ultimator of EI to $qv.v^{o}-\frac{q}{b}v^{2}v^{o}-\frac{2ql}{b}\times\overline{bv-vv}^{\frac{1}{2}}v^{o}+\frac{q}{b}ll\,v^{o}$, the Ultimator of the convex Solid about EI required. The Content therefore required muſt be equal to $\frac{1}{2}qv^{2}-\frac{q}{3b}v^{3}+\frac{q}{b}llv$ leſs $\frac{2ql}{b}$ multiplying the Subject of the Ultimator $\overline{bv-vv}^{\frac{1}{2}}v^{o}$. But it appears from the Anſwer to the ſeventh Example above, that the Subject of $\overline{bv-vv}^{\frac{1}{2}}v^{o}$ is $\frac{2}{3}b^{\frac{1}{2}}v^{\frac{3}{2}}$ leſs $\frac{v^{\frac{3}{2}}}{b^{\frac{1}{2}}}$ multiplying the Series $\frac{v}{5}+\frac{v^{2}}{28b}+\frac{v^{3}}{72b^{2}}+\frac{5v^{4}}{704b^{3}}+$ &c. Therefore the Content required muſt be equal to $qv^{2}\times\overline{\frac{1}{2}-\frac{v}{3b}}-qlv\times\sqrt{\frac{4v^{\frac{1}{2}}}{3b^{\frac{1}{2}}}-\frac{l}{b}}$ added to the Product of $qlv\times\frac{v^{\frac{1}{2}}}{b^{\frac{1}{2}}}$ multiplying the Series $\frac{2v}{5b}+\frac{v^{2}}{14b^{2}}+\frac{v^{3}}{36b^{3}}+\frac{5v^{4}}{352b^{4}}+\frac{7v^{5}}{832b^{5}}+\frac{7v^{6}}{1280b^{6}}+$ &c.

Example 16. Let it be required (See the Figure of the Cycloid, after Theo. 36.) to find the greater Area VMNPV of the common Cycloid, the Diameter of whoſe generating Circle is $VB=2c=b$, Sine $NP=s$, Abſciſſa $VP=v$, and correſpondent Arch of the Circle is $NV=u$.

Here, according to the known Property of the Circle, we have $s=\overline{bv-vv}^{\frac{1}{2}}$, whoſe Ultimator (Theo. 21.) is $\frac{cv^{o}-vv^{o}}{\overline{2cv-vv}^{\frac{1}{2}}}$; and the Ultimator of the correſpondent circular Arch NV (Theo. 34.)

is $\frac{cv^{\circ}}{\overline{2cv - vv}^{\frac{1}{2}}} = \frac{bv^{\circ}}{2 \times \overline{bcv - vv}^{\frac{1}{2}}}$. But in the Cycloid, the correſpondent Semiordinate $MP = OV$ has always the Part MN equal to the Length of the correſpondent circular Arch NV, and the other Part $NP = s$. Therefore (Theo. 18.) the Sum $\frac{cv^{\circ}}{\overline{2cv - vv}^{\frac{1}{2}}} + \frac{cv^{\circ} - vv^{\circ}}{\overline{2cv - vv}^{\frac{1}{2}}} = \frac{bv^{\circ} - vv^{\circ}}{\overline{bv - vv}^{\frac{1}{2}}}$ muſt be the Ultimator of the Cycloidal Semiordinate $MP = OV$. Whence (Theo. 53.) we have $1 : v = \frac{bv^{\circ} - vv^{\circ}}{\overline{bv - vv}^{\frac{1}{2}}} : \frac{bv - vv}{\overline{bv - vv}^{\frac{1}{2}}} v^{\circ}$ the Ultimator of the concave Area OVM equal to $\overline{bv - vv}^{\frac{1}{2}} v^{\circ}$. Now for finding the Ultimator of the convex circular Area NVP, we have (Theo. 52.) the Proportion, as Unity is to the Semiordinate $s = \overline{bv - vv}^{\frac{1}{2}}$; ſo is v° to $\overline{bv - vv}^{\frac{1}{2}} v^{\circ}$. Whence it appears, that the Ultimators of the concave Cycloidal Area OVM, and its correſpondent convex circular Area NVP are always equal, and conſequently, ſo muſt the Areas themſelves. But the Area of the Parallelogram $MOVP$, which is made up of the Sum of theſe two Areas with the included mixed Area MVN is equal to $\overline{MN + NP} \times VP$ or $\overline{u + s} \times v$. Alſo the Area MVP is equal to the Area of the Sector $NCVN$ leſs the Area of the right-angled Triangle $NCPN$, or to $\frac{1}{2} cu - \frac{c - v}{2} s$. Therefore the Areas OMV and NVP are together equal to $cu - cs + vs$, which Sum being ſubtracted out of the Parallelogram $MOVP$ or $uv + vs$ muſt leave $uv + cs - cu$ or $uv - c \times \overline{u - s}$ equal to the included mixed Area MVN. Conſequently, the whole convex Cycloidal Area $VMNPV$ required, is equal to $vu - \frac{1}{2} cu + \frac{1}{2} vs - \frac{1}{2} cs$.

Hence, if $v = 2c = VP$, whence $VNB = AB = u = \frac{1}{2} q$ or half the Circumference of the generating Circle, and $s = 0$.

Then we have the concave Area QVMA equal to the Area of the generating Semicircle $= \frac{1}{4}qc$; and the included mixed Area AMVNB or $uv + cs - cu = \frac{1}{2}q \times 2c + c \times o - \frac{1}{2}qc = \frac{1}{2}qc$ equal to the Area of the whole generating Circle.

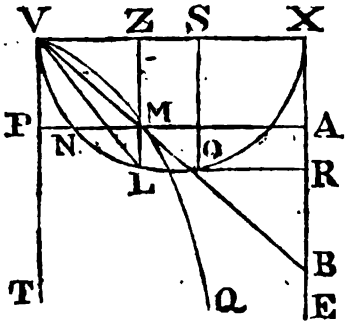
Example 17. Let it be required (ſee the ſame Figure as in the laſt) to find the Content of the Solid, which is formed by the Rotation of the greater cycloidal Area MVPM about the Semiordinate MP, all Things being repreſented, as in the laſt.

Here the Content is the ſame with that of the convex Solid M (Theo. 62.). If therefore, we ſubſtitute $\frac{b-v}{\overline{bv-vv}^{\frac{1}{2}}}v^{o}$ the Ultimator of MP for the s^{o} there, we have $\frac{2q\dot{v}}{b} \times \frac{bv-vv}{\overline{bv-vv}^{\frac{1}{2}}}v^{o} - \frac{q\dot{v}}{b} \times \frac{bv-vv}{\overline{bv-vv}^{\frac{1}{2}}}v^{o} = \frac{2q\dot{v}}{b} \times \overline{bv-vv}^{\frac{1}{2}}v^{o} - \frac{q}{b} \times \overline{bv-vv}^{\frac{1}{2}}vv^{o}$ the Ultimator of the circumambient Solid formed at the ſame Time by the Rotation of the concave Area MOVM about the ſame Axis MP. But the Subject of the Ultimator $\overline{bv-vv}^{\frac{1}{2}}v^{o}$ (ſee the Anſwer to the ſeventh Example is $\frac{2}{3}b^{\frac{1}{2}}v^{\frac{1}{2}}v$ leſs the Product of $v.\frac{v^{\frac{1}{2}}}{b^{\frac{1}{2}}}$ multiplying the Series $\frac{v}{5} + \frac{v^2}{28b} + \frac{v^3}{72b^2} + \frac{5v^4}{704b^3} + \frac{7v^5}{1664b^4} + \frac{7v^6}{2560b^5} +$ &c. And conſequently the Subject of $\overline{bv-vv}^{\frac{1}{2}}vv^{o}$ (as this and no other Difference from the other, but the Multiplication of the Whole by v) muſt be $\frac{2}{5}b^{\frac{1}{2}}v^{\frac{1}{2}}v^2$ leſs the Product of $vv \times \frac{v^{\frac{1}{2}}}{b^{\frac{1}{2}}}$ multiplying the Series $\frac{v}{7} + \frac{v^2}{36b} + \frac{v^3}{88b^2} + \frac{5v^4}{832b^3} + \frac{7v^5}{1920b^4} + \frac{21v^6}{8704b^5} +$ &c. The former Subject therefore being multiplied by $\frac{2qv}{b}$ is equal to $\frac{4qv^2v^{\frac{1}{2}}}{3b^{\frac{1}{2}}}$ leſs the Product of $\frac{qv^2v^{\frac{1}{2}}}{b^{\frac{1}{2}}}$ multiplying the Series $\frac{2v}{5b} + \frac{v^2}{14b^2} + \frac{v^3}{36b^3} + \frac{5v^4}{352b^4} + \frac{7v^5}{832b^5} + \frac{7v^6}{1280b^6} +$ &c. For the the Whole of which put X. And the latter Subject being multiplied by $\frac{q}{b}$ is equal to $\frac{2qv^2v^{\frac{1}{2}}}{5b^{\frac{1}{2}}}$ leſs the Product of $\frac{qv^2v^{\frac{1}{2}}}{b^{\frac{1}{2}}}$ multiplying the Series $\frac{v}{7b} + \frac{v^2}{36b^2} + \frac{5v^4}{832b^4} + \frac{7v^5}{1920b^5} + \frac{21v^6}{8704b^6} +$ &c. For the Whole of which put Z. Then (Theo. 59.) the Subject Z muſt be the Content of the Concave Solid V, which is formed by the Rotation of the leſſer cycloidal Area OMVO about the Axis OV. And this added to the circumſcribing Cylinder $\frac{qvv}{b} \times MP = \frac{qv^2u}{b} + \frac{qv^2s}{b}$ makes $\frac{qv^2}{b} \times \overline{u+s} + Z$; from whence X being ſubtracted (Theo. 60.) we have the Content required.

If therefore $v = b = VB$; whence $u = \frac{1}{2}q$, and $s = 0$; then we have $Z = \frac{2}{5}qb^2$ leſs the Product of qb^2 multiplied by the Series $\frac{1}{7} + \frac{1}{36} + \frac{1}{88} + \frac{5}{832} + \frac{7}{1920} + \frac{21}{8704} +$ &c. or qb^2 multiplied by the Series $\frac{2}{5} - \frac{1}{7} - \frac{1}{36} - \frac{1}{88} - \frac{5}{832} - \frac{7}{1920} - \frac{21}{8704} -$ &c. equal to the Content of the Solid, which is formed by the Rotation of the cycloidal Area AQVMA about the Axis QV. And for $\frac{qvv}{b} \times \overline{u+s} + Z - X$, we have the Sum of $\frac{1}{2}q^2b$ added to the Product of qb^2 multiplied by the Series $\frac{2}{5}$

$-\frac{1}{7}-\frac{1}{36}-\frac{1}{88}-\frac{5}{832}-\frac{7}{1920}-\frac{21}{8704}-$ &c. From whence ſubtracting the Product of qb^2 multiplied by the Series $\frac{4}{3}-\frac{2}{5}-\frac{1}{14}-\frac{1}{36}-\frac{5}{352}-\frac{7}{832}-\frac{7}{1280}-$ &c. we have the Content of the Solid, which is formed by the Rotation of the greater Cycloidal Area A M V B A about the Axis A B.

Example 18. Let it be required to determine the leſſer Area of the * Ciſſoid of *Diocles*; which is contained within the Diameter $VX = b$ of the generating Circle, the Curve V M Q &c. and the adjacent Aſymptote X A B E &c. each produced *ad infinitum*.



Here (putting $PV = v$, and $PM = s$) the Equation of the Curve is $\frac{sss}{b-s} = vv$. Whence $\frac{s^{\frac{3}{2}}}{\overline{b-s}^{\frac{1}{2}}} = v$ or $\frac{ss}{\overline{bs-ss}^{\frac{1}{2}}} = v$. Conſequently, (Theo. 51.) $\frac{s^2 s^0}{\overline{bs-ss}^{\frac{1}{2}}}$ will be the Ultimator of the leſſer Area V M Z. But ss being the ſame with $bs - \overline{bs-ss}$, therefore the ſame Ultimator may be expreſſed by $\frac{bss^0}{\overline{bs-ss}^{\frac{1}{2}}} - \frac{\overline{bs-ss}}{\overline{bs-ss}^{\frac{1}{2}}} s^0 = \frac{bss^0}{\overline{bs-ss}^{\frac{1}{2}}} - \overline{bs-ss}^{\frac{1}{2}} s^0$. And if we put s for the Abſciſſa V Z of a Circle, whoſe Diameter is $b = VX$, and conſequently, its Semiordinate $ZL = \overline{bs-ss}^{\frac{1}{2}}$, we have (Theo. 50.) $\overline{bs-ss}^{\frac{1}{2}} s^0$ equal to the Ultimator of the greater circular Area VNLZ.

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* *See* Ciſſoid *in the Appendix.*

V N L Z. Again, the Area of the Triangle V L Z is equal to $\frac{1}{2}$ V Z × Z L $= \frac{1}{2}\, s \times \overline{bs - ss}^{\frac{1}{2}}$, whoſe Ultimator $\frac{3bs - 4ss}{4 \times \overline{bs - ss}^{\frac{1}{2}}} s^{\circ}$ ſubtracted from the laſt Ultimator $\overline{bss - ss}^{\frac{1}{2}}\, s^{\circ}$ leaves $\frac{bss^{\circ}}{4 \times \overline{bs - ss}^{\frac{1}{2}}}$ equal (Theo. 17.) to the Ultimator of the circular Segment V N L V, or to one fourth Part of the Ultimator $\frac{bss^{\circ}}{\overline{bs - ss}^{\frac{1}{2}}}$. Therefore, every ſuch concave Area V M Z in the Ciſſoid is (Theo. 17.) equal to the Remainder which is left after the Subtraction of its correſpondent circular Area V N L Z out of four Times the Area of the circular Segment V N L V, which anſwers to it.

But when the leſſer ciſſoidal Area is that between the Curve and its Aſymptote, each produced infinitely; then it is plain that Z L vaniſhes to nothing; and the circular Area V N L Z becomes all one with the Segment V M L V, and each the ſame with the Semi-circle V N L X V. Conſequently, the leſſer Area of the Ciſſoid required is equal to three Times the generating Semi-circle.

Example 19. The ſame Things being again ſuppoſed as before, let it be required to find the Content of the Solid, which is formed by the Rotation of the ſame infinite ciſſoidal Space about its Aſymptote X E.

Becauſe $v = \frac{s^{\frac{3}{2}}}{\overline{b - s}^{\frac{1}{2}}}$, therefore the circumſcribing Cylinder which is formed by the Rotation of the greater Area V M P V about V P will be $\frac{q}{b}\, ssv = \frac{qs^{\frac{7}{2}}}{b \times \overline{b - s}^{\frac{1}{2}}} = \frac{qs^{\frac{7}{2}}}{b^{\frac{3}{2}}}$ multiplying the Series $1 + \frac{s}{2b} + \frac{3s^2}{8b^2} + \frac{5s^3}{16b^3} + \frac{35s^4}{128b^4} + \frac{63s^5}{256b^5} + \frac{231s^6}{1024b^6} +$ &c.

and $v^{\circ} = \overline{\frac{3}{2} b s^{\frac{1}{2}} - s^{\frac{3}{2}}} \times \frac{s^{\circ}}{\overline{b-s}^{\frac{1}{2}}} = \overline{\frac{3}{2} b s^{\frac{1}{2}} - s^{\frac{3}{2}}} \times s^{\circ}$ multiplying the Series $\frac{1}{b^{\frac{3}{2}}} + \frac{3s}{2b^{\frac{5}{2}}} + \frac{15s^2}{8b^{\frac{7}{2}}} + \frac{35s^3}{16b^{\frac{9}{2}}} + \frac{315s^4}{128b^{\frac{11}{2}}} + \frac{693s^5}{256b^{\frac{13}{2}}} + \frac{3003s^6}{1024b^{\frac{15}{2}}} +$ &c. $= \frac{s^{\frac{1}{2}}}{b^{\frac{1}{2}}} s^{\circ}$ multiplying the Series $\frac{3}{2} + \frac{5s}{4b} + \frac{21s^2}{16b^2} + \frac{45s^3}{32b^3} + \frac{385s^4}{256b^4} + \frac{819s^5}{512b^5} + \frac{3465s^6}{2048b^6} +$ &c. The Ultimator therefore of the convex Solid V (Theo. 57.) will be $\frac{q}{b} s^2 v^{\circ} = \frac{qs^{\frac{5}{2}}}{b^{\frac{3}{2}}} s^{\circ}$ multiplying the Series $\frac{3}{2} + \frac{5s}{4b} + \frac{21s^2}{16b^2} + \frac{45s^3}{32b^2} + \frac{385s^4}{256b^4} + \frac{819s^5}{512b^5} + \frac{3465s^6}{2048b^6} +$ &c. Whoſe Subject is $\frac{qs^{\frac{7}{2}}}{b^{\frac{3}{2}}}$ multiplying the Series $\frac{3}{7} + \frac{5s}{18b} + \frac{21s^2}{88b^2} + \frac{45s^3}{208b^3} + \frac{385s^4}{1920b^4} + \frac{819s^5}{4352b^5} + \frac{3465s^6}{19456b^6} +$ &c. Which being added to the circumſcribing Cylinder makes the Sum equal to $\frac{qs^{\frac{7}{2}}}{b^{\frac{3}{2}}}$ multiplying the Series $\frac{10}{7} + \frac{7s}{9b} + \frac{27s^2}{44b^2} + \frac{55s^3}{104b^3} + \frac{91s^4}{192b^4} + \frac{945s^5}{2176b^5} + \frac{3927s^6}{9728b^6} +$ &c. Next (Theo. 64.) the Ultimator $\frac{2q}{b} \dot{s} s v^{\circ}$ equals $\frac{q\dot{s}s^{\frac{3}{2}} s^{\circ}}{b^{\frac{3}{2}}}$ multiplying the Series $3 + \frac{5s}{2b} + \frac{21s^2}{8b^2} + \frac{45s^3}{16b^3} + \frac{385s^4}{128b^4} + \frac{819s^5}{256b^5} + \frac{3465s^6}{1024b^6} +$. Whoſe Subject is equal to $\frac{qs^{\frac{7}{2}}}{b^{\frac{3}{2}}}$ multiplying the Series $\frac{6}{5} + \frac{5s}{7b} + \frac{7s^2}{12b^2} + \frac{45s^3}{88b^3} + \frac{385s^4}{832b^4} + \frac{819s^5}{1920b^5} + \frac{3465s^6}{8704b^6} +$ &c. Which ſubtracted from the foregoing Sum leaves $\frac{qs^{\frac{7}{2}}}{b^{\frac{3}{2}}}$ multiplying the Series $\frac{8}{35}+\frac{4s}{63b}+\frac{s^2}{33b^2}+\frac{5s^3}{286b^3}+\frac{7s^4}{624b^4}+\frac{21s^5}{2720b^5}+\frac{231s^6}{41344b^6}$ for the Content of every concave Solid M, formed by the Rotation of the leſſer ciſſoidal Area VMZV about the Axis ZM. Conſequently the Content required, when s is the ſame with b, muſt be equal to qbb multiplied by the Series $\frac{8}{35}+\frac{4}{63}+\frac{1}{33}+\frac{5}{286}+\frac{7}{624}+\frac{21}{2720}+\frac{231}{41344}+$ &c.

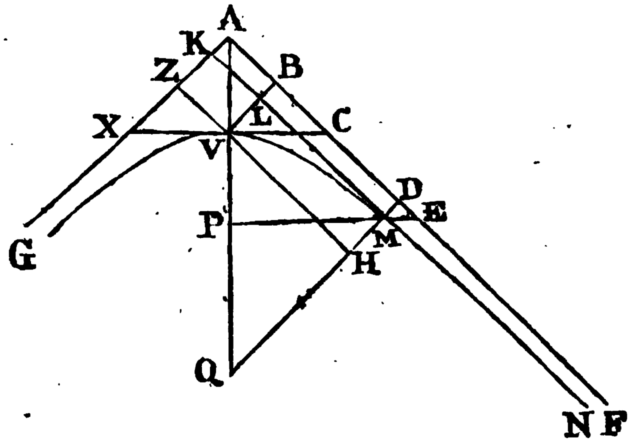
Example 20. Let it be required to find the concave Superficies of the laſt Ciſſoidal Content.

Here for the Length of the Arch ſay, As the Ultimator $\frac{s^{\circ}}{\overline{b-s}^{\frac{3}{2}}}\times\sqrt{bs^{\frac{1}{2}}-s^{\frac{3}{2}}}$ of the Abſciſſa VP is to the Ultimator s° of the Semiordinate PM; ſo is the Semiordinate s to the Subnormal $\frac{\overline{b-s}^{\frac{3}{2}}\times s^{\frac{1}{2}}}{\frac{3}{2}b-s}$. Whence the Normal is $\frac{b\times\overline{bs-\frac{3}{4}ss}^{\frac{1}{2}}}{\frac{3}{2}b-s}$. Then, as the Subnormal is to the Normal; or as $\overline{b-s}^{\frac{3}{2}}$ is to $b\times\overline{b-\frac{3}{4}s}^{\frac{1}{2}}$, ſo is the Ultimator s° of the Semiordinate to the Ultimator $\frac{bs^{\circ}\times\overline{b-\frac{3}{4}s}^{\frac{1}{2}}}{\overline{b-s}^{\frac{3}{2}}}$ $=\frac{bs^{\circ}\times\overline{4b-3s}^{\frac{1}{2}}}{2\times\overline{b-s}\times\overline{b-s}^{\frac{1}{2}}}$ of the Arch VM. Laſtly, (Theo. 64.) as $\frac{1}{2}b$ is to qMA or $q\times\overline{b-s}$, ſo is the Ultimator $\frac{bs^{\circ}\times\overline{4b-3s}^{\frac{1}{2}}}{2\times\overline{b-s}\times\overline{b-s}^{\frac{1}{2}}}$, to the Ultimator $\frac{qs^{\circ}\times\overline{4b-3s}^{\frac{1}{2}}}{\overline{b-s}^{\frac{1}{2}}}$ of the concave Superficies required.

But $\overline{4b-3s}^{\frac{1}{2}}$ equals $2b^{\frac{1}{2}}-\frac{3s}{4b^{\frac{1}{2}}}-\frac{9s^2}{64b^{\frac{3}{2}}}-\frac{27s^3}{512b^{\frac{5}{2}}}-\frac{405s^4}{16384b^{\frac{7}{2}}}-$ &

— &c. and $\overline{b-s}^{\frac{1}{2}}$ equals $b^{\frac{1}{2}}-\frac{s}{2b^{\frac{1}{2}}}-\frac{s^2}{8b^{\frac{1}{2}}}-\frac{s^3}{16b^{\frac{5}{2}}}-\frac{5s^4}{128b^{\frac{7}{2}}}$ — &c. and the Quotient (multiplied by s^0) of the former divided by the latter is $2s^0+\frac{ss^0}{4b}+\frac{15s^2s^0}{64b^2}+\frac{113s^3s^0}{512b^3}+\frac{113s^4s^0}{1024b^4}$ + &c. Whoſe Subject multiplied by q, equals 2qs multiplying the Series $1+\frac{s}{16b}+\frac{5s^2}{128b}+\frac{113s^3}{4096b^3}+\frac{113s^4}{10240b^4}$ + &c. Conſequently, when s is the ſame with b, we have 2 q b multiplied by the Series $1+\frac{1}{16}+\frac{5}{128}+\frac{113}{4096}+\frac{113}{10240}$ + &c. for the concave Superficies required.

Example 21. In the annexed * Rectangular Hyperbola, let it be required to determine the Area, which is contained between the Semicurve VMN &c. and its adjacent Aſymptote BCDF &c. ſuppoſing each to be produced *ad infinitum.*



The rectangular Hyperbola, is ſuch a one as has for its Aſymptotes the Legs of the Rectangle GAF, and conſequently, its Semitranſverſe Diameter AV being the ſame with its Semi-conjugate VC = c (putting VP = v, and PM = s) the Equation of the Curve is $2cv+vv=ss$. Whence $v=\overline{cc+ss}^{\frac{1}{2}}-c$, or (putting $AB=BV=\frac{c}{2^{\frac{1}{2}}}=n$) $v=\overline{2nn+ss}^{\frac{1}{2}}-2^{\frac{1}{2}}n$.

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* *See* Rectangular Hyperbola *in the Appendix.*

Produce PM to meet the Aſymptote in E, and through M draw QD parallel to GA and LM parallel to AD. Alſo through V draw ZH parallel to AD; and put $z = HM = VL$. Whence, becauſe all the Triangles here are ſimilar, having equal Legs about the Rightangle in each, we have $AB : AV = PM : QM$, or $n : 2^{\frac{1}{2}}n = s : 2^{\frac{1}{2}}s = QM$. Alſo becauſe $VP + PQ = v + s = \overline{2nn+ss}^{\frac{1}{2}} - 2^{\frac{1}{2}}n + s = VQ$, and $2^{\frac{1}{2}}n : n = VQ : QH$; therefore QH equals $\frac{s + \overline{2nn+ss}^{\frac{1}{2}}}{2^{\frac{1}{2}}} - n = VH =$ the Abſciſſa of the greater Area VMHV, whoſe Semiordinate is $QM - QH = n + \frac{s - \overline{2nn+ss}^{\frac{1}{2}}}{2^{\frac{1}{2}}} = z = HM$. Conſequently, $s = \frac{2nz - zz}{2^{\frac{1}{2}} \times \overline{n-z}}$, and $VH = \frac{s + \overline{2nn+ss}^{\frac{1}{2}}}{2^{\frac{1}{2}}} - n = \frac{nz}{n-z}$. Conſequently, for the leſſer Area VMLV, ſince the Ultimator of $HM = z$ is $\dot{z}$, we have (Theo. 53.) $1 : \frac{nz}{n-z} = \dot{z} : \frac{nz\dot{z}}{n-z} = z\dot{z} + \frac{z^2\dot{z}}{n} + \frac{z^3\dot{z}}{n^2} + \frac{z^4\dot{z}}{n^3} + \frac{z^5\dot{z}}{n^4} + \frac{z^6\dot{z}}{n^5} +$ &c. Whoſe Subject is z^2 multiplying the Series $\frac{1}{2} + \frac{z}{3n} + \frac{z^2}{4n^2} + \frac{z^3}{5n^3} + \frac{z^4}{6n^4} + \frac{z^5}{7n^5} + \frac{z^6}{8n^6} +$ &c. Whence, by the infinite Production of MN and DF, at which Time VL muſt be equal to VB, or $z = n$, we have the ſame Subject equal to nn or $\overline{VB}^2$ multiplied by the Series $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} +$ &c. for the Area required.

Here I cannot but obſerve the groſs Miſtake, which the Writers of Fluxions have fallen into in the Anſwer of this very Queſtion. *See* Mr. Hodgſon's Fluxions, *Part* 3, *Sect.* II. *Ex*, 6. *Page* 276. There, after having put AB = 1, ZH = x, and MD = LB = y, the Ultimator of the Area required, is found to be $x^{-1} x^{\circ}$; an Ultimator, which by both their and our Principles (Theo. 20, 24.) muſt have the Subject equal to 0 or Nothing. But yet this Ultimator of Nothing, is unaccountably forced to have the Subject x^{-1+1} or $\frac{x}{x} = 1 = yx$. By which manifeſt Fallacy, they have found Means to determine the Area above to be roundly equal to $\overline{VB}^{2}$ or nn.

Example 22. The ſame Things being ſuppoſed as in the laſt, let it be required to find the Content of the Solid, which is formed by the Rotation of the rectangular Hyperbolic Space VB, MD, NF, &c. about the adjacent Aſymptote BF &c.

Here in the leſſer Area VLMV we have HM = z and $VH = \frac{nz}{n-z}$. Whence (Theo. 63.) for $\frac{2q\dot{s}}{b} s v^{\circ} - \frac{q}{b} ss v^{\circ}$ (by ſubſtituting $\dot{z}$ for s, $\frac{nz}{n-z}$ for v, and $\frac{nnz^{\circ}}{nn-2nz+zz}$ for v°) we have $\frac{2q\dot{z}}{b} \times \frac{n^2 zz^{\circ}}{n^2-2nz+z^2} - \frac{q}{b} \times \frac{n^2 z^2 z^{\circ}}{n^2-2nz+zz}$ equal to the Ultimator of the Solid, which is formed by the Rotation of the greater Area VHMV about the Axis LM. But $\frac{n^2 zz^{\circ}}{n^2-2nz+zz}$ is equal to $zz^{\circ} + \frac{2z^2 z^{\circ}}{n} + \frac{3z^3 z^{\circ}}{n^2} + \frac{4z^4 z^{\circ}}{n^3} + \frac{5z^5 z^{\circ}}{n^4} + \frac{6z^6 z^{\circ}}{n^5} +$ &c. Whoſe Subject multiplied by $\frac{2q\dot{z}}{b}$ is equal to $\frac{q}{b} z^2$ multiplying the Series

z 1 +

$2+\frac{4z}{3n}+\frac{6z^2}{4n^2}+\frac{8z^3}{5n^3}+\frac{10z^4}{6n^4}+\frac{12z^5}{7n^5}+\frac{14z^6}{8n^6}+$ &c. And after the ſame Manner, the Subject of the Ultimator $\frac{q}{b}\times\frac{n^2z^2z^0}{n^2-2nz+z^2}$ will be found equal to $\frac{q}{b}z^3$ multiplying the Series $\frac{1}{3}+\frac{2z}{4n}+\frac{3z^2}{5n^2}+\frac{4z^3}{6n^3}+\frac{5z^4}{7n^4}+\frac{6z^5}{8n^5}+\frac{7z^6}{9n^6}+$ &c. equal to the Content of the Solid which is formed (Theo. 57.) by the Rotation of the greater Area V H M V about the Abſciſſa V H. Which latter Subject added to the circumſcribing Cylinder $\frac{qnz^2}{b\times\overline{n-z}}$, or to $\frac{q}{b}z^3$ multiplying the Series $1+\frac{z}{n}+\frac{z^2}{n^2}+\frac{z^3}{n^3}+\frac{z^4}{n^4}+$ &c. makes the Sum equal to $\frac{q}{b}z^3$ multiplying the Series $\frac{4}{3}+\frac{6z}{4n}+\frac{8z^2}{9n^2}+\frac{10z^3}{6n^3}+\frac{12z^4}{7n^4}+\frac{14z^5}{8n^5}+$ &c. From which Sum the former Subject being ſubtracted (Theo. 64.) leaves $\frac{q}{b}z^3$ multiplying the Series $\frac{2}{3}+\frac{z}{6n}+\frac{z^2}{10n^2}+\frac{z^3}{15n^3}+\frac{z^4}{21n^4}+\frac{z^5}{28n^5}+$ &c. or $\frac{q}{b}z^3$ multiplying the Series $\frac{1}{3}+\frac{z}{\overline{3+3}\times z}+\frac{z^2}{\overline{6+4}\times n^2}+\frac{z^3}{\overline{10+5}\times n^3}+\frac{z^4}{\overline{15+6}\times n^4}+\frac{z^5}{\overline{21+7}\times n^5}+$ &c. for the Content of the Solid formed by the Rotation of the leſſer Area V L M V about the Axis L M. But when the Axis L M coincides with the Aſymptote B F &c. it is plain that V L muſt be the ſame with V B or $z=n$. Conſequently, the Content required muſt be equal to $\frac{q}{b}n^2$ multiplied by the Series $\frac{1}{3}+\frac{1}{3+3}+\frac{1}{6+4}+\frac{1}{10+5}+\frac{1}{15+6}+\frac{1}{21+7}+$ &c.

Example 23. Let it be required to find the Concave Superficies of the laſt Hyperbolic Content.

Here put $MD=y=n-z$; whence $z=HM=n-y$, and $VH=\frac{nz}{n-z}=\frac{nn}{y}-n$. Then for the Length of the Arch VM ſay. As the Ultimator $\frac{n^2y^0}{yy}$ of the Abſciſſa VH is to the Ultimator y^0 of the Semiordinate HM; ſo is the Semiordinate $n-y$ to the Subnormal $\frac{ny^2-y^3}{nn}$. Whence the Normal is $\frac{n-y}{n^2}\times\overline{n^4+y^4}^{\frac{1}{2}}$. Next, as the

ſo is y^0 to $\frac{y^0}{y^2}\overline{n^4+y^4}^{\frac{1}{2}}$ the Ultimator

Subnormal is to the Normal, or as y^2 is to $\overline{n^4+y^4}^{\frac{1}{2}}$ of the Arch VM. Laſtly (Theo. 64.) as $\frac{1}{2}b$ is to qy, ſo is $\frac{y^0}{y^2}\times\overline{n^4+y^4}^{\frac{1}{2}}$ to $\frac{2qy^0}{by}\times\overline{n^4+y^4}^{\frac{1}{2}}$ the Ultimator of the concave Superficies required. But $\overline{n^4+y^4}^{\frac{1}{2}}$ equals $n^2+\frac{y^4}{2n^2}-\frac{y^8}{8n^6}+\frac{y^{12}}{16n^{10}}-\frac{5y^{16}}{128n^{14}}+\frac{7y^{20}}{256n^{18}}-\frac{21y^{24}}{1024n^{22}}+$ &c. Therefore the ſame Ultimator is equal to $\frac{2q}{b}$ multiplying the Series $\frac{n^2y^0}{y}+\frac{y^3y^0}{2n^2}-\frac{y^7y^0}{8n^6}+\frac{y^{11}y^0}{16n^{10}}-\frac{5y^{15}y^0}{128n^{14}}+\frac{7y^{19}y^0}{256n^{18}}-\frac{21y^{23}y^0}{1024n^{22}}+$ &c. The Subject of all which, excepting $\frac{2qn^2y^0}{by}$ equals $\frac{2q}{b}$ multiplying the

Series $\frac{y^{4}}{8n^{2}} - \frac{y^{8}}{64n^{6}} + \frac{y^{12}}{192n^{10}} - \frac{5y^{16}}{2048n^{14}} + \frac{7y^{20}}{5120n^{18}} + \frac{7y^{24}}{8192n^{22}}$ + &c. Which, when y (as here) becomes the ſame with n, makes $\frac{q}{b}n^{2}$ multiplied by the Series $\frac{1}{4} - \frac{1}{32} + \frac{1}{96} - \frac{5}{1024} + \frac{7}{2560} - \frac{7}{4096} +$ &c. $= X$. To this we muſt add the Subject of $\frac{2qn^{4}y^{\circ}}{y}$, which (as has been ſhewn) can in itſelf have no Ultimator: But by reſtoring $n - z$ for y and $-z^{\circ}$ for y°, the ſame Ultimator becomes $\frac{-2qn^{2}z^{\circ}}{b \times \overline{n - z}} = -\frac{2q}{b}$ multiplying the Series $nz^{\circ} + zz^{\circ} + \frac{z^{2}z^{\circ}}{n} + \frac{z^{3}z^{\circ}}{n^{2}} + \frac{z^{4}z^{\circ}}{n^{3}} + \frac{z^{5}z^{\circ}}{n^{4}} + \frac{z^{6}z^{\circ}}{n^{5}} +$ &c. Whoſe Subject, after z is made equal to n, gives $-\frac{2q}{b}n^{2}$ multiplied by the Series $\frac{3}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} +$ &c. $= -Y$. Conſequently, the Superficies required, is equal to $+Y - X$; and not to $\frac{2q}{b}nn + X$, as has been hitherto falſely taught.

Example 24. Let it be required to find the Content of the Ambient Solid, which is formed by the Rotation (ſee the laſt Figure) of the leſſer Hyperbolic Area V L M V about the Part Z K of the oppoſite Aſymptote A G.

Here, as before, we have $VL = HM = z$, $ZV = VB = n$, and $VH = LM = \frac{nz}{n - z}$. Whence, the Cylinder formed at the ſame Time by the Rotation of the Rectangle Z V L K about

ZK is $\frac{q}{b}nnz$, the Cylinder formed by the Rotation of the Rectangle ZHMK about ZK is $\frac{q}{b}\times\frac{n^4z}{\overline{n-z}^2}$, and conſequently the cylindrical Ambient formed at the ſame Time by the Rotation of the Rectangle VHML is $\frac{q}{b}\times\frac{n^4z}{\overline{n-z}^2}-\frac{q}{b}nnz=\frac{q}{b}\times\frac{2n^3z^2-n^2z^3}{\overline{n-z}^2}$ or $\frac{q}{b}n^2z^2\times\frac{2n-z}{\overline{n-z}^2}$. Therefore, the Ultimator (Theo. 67.) $\frac{qs^{\circ}}{b}\times\overline{1+v}^2$ (by putting z for s, n for l, and $\frac{nz}{n-z}$ for v) equal to $\frac{q}{b}n^2\times\frac{n^2z^{\circ}}{\overline{n-z}^2}$, is the Ultimator of the Solid Content formed at the ſame Time by the Rotation of the mixed Area ZVLK + VLMV. But $\frac{n^2z^{\circ}}{\overline{n-z}^2}=z^{\circ}+\frac{2zz^{\circ}}{n}+\frac{3z^2z^{\circ}}{n^2}+\frac{4z^3z^{\circ}}{n^3}+\frac{5z^4z^{\circ}}{n^4}+\frac{6z^5z^{\circ}}{n^5}+$ &c. Whoſe Subject is $z+\frac{z^2}{n}+\frac{z^3}{n^2}+\frac{z^4}{n^3}+\frac{z^5}{n^4}+\frac{z^6}{n^5}+$ &c. $=\frac{nz}{n-z}$. Conſequently, the Solid formed by the Rotation of the mixed Area ZVLK + VLMV is equal to $\frac{q}{b}n^2\times\frac{nz}{n-z}=\frac{qn^3z}{b\times\overline{n-z}}$. But $\frac{qn^3z}{b\times\overline{n-z}}\times\frac{n}{n-z}=\frac{q}{b}\times\frac{n^4z}{\overline{n-z}^2}$ the circumſcribing Cylinder. Therefore its circumſcribing Cylinder $\frac{q}{b}\times\frac{n^4z}{\overline{n-z}^2}$ is always to the ſaid mixed Solid $\frac{qn^3z}{b\times\overline{n-z}}$ as n to n—z, or as VB is to LB. Hence, that mixed Solid $\frac{qn^3z}{b\times\overline{n-z}}$

made less by the Cylinder $\frac{q}{b}nnz$ gives $\frac{qn^2z^2}{b\times\overline{n-z}}$ for the Content of the ambient Solid required. That is, as n or K L is to $\frac{nz}{n-z}$ or LM, ſo is the Cylinder $\frac{q}{b}nnz$ (whoſe Radius is $KL=n$, and Height is $VL=z$) to $\frac{qn^2z^2}{b\times\overline{n-z}}$ the correſpondent ambient Solid required. Otherwiſe, multiplying this ambient Solid by $\frac{n-z}{n-z}$, the ſame may be expreſſed by $\frac{q}{b}\times\frac{n^3z^2-n^2z^3}{\overline{n-z}^2}$ Which ſhews that its circumſcribing ambient Cylinder $\frac{q}{b}\times\frac{2n^3z^2-n^2z^3}{\overline{n-z}^2}$ is is to it, as $2n-z$ is to $n-z$, or (by ſubſtituting $y=MD$ for $n-z$) as $n+y$ is to y.

I have dwelt longer upon the Anſwer to this Queſtion, to ſhew the groſs Abſurdities into which Men are led, by blindly giving up their Underſtandings to thoſe Chimeras of Fluxions by which they can perſuade themſelves, that they are able to do Impoſſibilities. A notorious Inſtance of which is to be ſeen in Mr. *Hodgſon*'s Fluxions, Page 352: Where it is pretended to determine the Content of the Solid formed by the Rotation of the Hyperbolic Space V B, M D, N F, &c. about the oppoſite Aſymptote Z A. A Taſk as impoſſible to be performed, as for a Man to confine the whole Univerſe within the Compaſs of a Nut-ſhell. But notwithſtanding, after a ſurpriſing Manner, that infinite Content is in that Place found equal to no more, if I underſtand the Gentleman rightly, than $\frac{nnq}{-2}$; or (putting $n=$ Unity) then $-\frac{q}{2}$. And yet it is plain to any one of an ordinary Capacity, that (for Example) if $KM=AD=\frac{nn}{n-z}=b$

L =

$\frac{1}{1-z} = 1000000$. (Whence $z = VL = 0.999999$, and $y =$ KA $= 0.000001$) we have (making $b = 2n = 2$) the Content of the right Cylinder formed by the Rotation of KMDAK about the Axis AK equal to $\frac{q}{2} \times 1000000 = 500000q$. All which and an infinite Quantity more, muſt be contained within $-\frac{1}{2}q$. If Fluxions can do this, I muſt acknowledge them much above my Comprehenſion.

The Truth is, whenever an Abſciſſa, $VH = \frac{nz}{n-z}$ in the Hyperbola, or $VP = \frac{s^{\frac{3}{2}}}{\overline{b-s}^{\frac{1}{2}}}$ in the Ciſſoid (ſee their reſpective Figures before); where the Diviſors $n-z$ and $b-s$ decreaſe *ad infinitum* as z and s encreaſe: Then in finding the Contents of the Solids formed by the Rotation of the Rectangle ZHMK about ZK to the Hyperbola; or the Rectangle VPMZ about VZ to the Ciſſoid; if we be obliged to expreſs the final Anſwer by an infinite Series, the greater z or s are, the more always ſhall its reſpective Series err from the Truth. So that in that Caſe we muſt expreſs the Semiordinate by the Abſciſſa; and not, as in the foregoing Examples, the Abſciſſa by the Semiordinate. And conſequently, every Aſymptotical Solid formed this Way, muſt be as infinitely great, as the Abſciſſa itſelf is infinitely long.

What remains concerning other Sorts of Curves, I refer to the Appendix, where I ſhall treat of them more methodically. In the mean Time obſerve, that I have all along purpoſely ommitted treating of them under univerſal Exponents, as a needleſs Puzzling of the Learner: Since the Solution of any one Particular, is a plain Diſcovery how we are to proceed in every other of the ſame Claſſis; which ſhall be fully ſhewn hereafter.

THE
APPENDIX.
Concerning the Nature and Deſcription of *Curves.*

I.

Of the Kinds of Curves *in general.*

EVERY Line may be conſidered, as generated by the Motion of an indiviſible Point; and if the Motion of ſuch generating Point aim always at the ſame immoveable Point, it is ſaid to keep the ſame Direction; but if it aim at firſt one Point and then another, it is ſaid to change its Direction.

A Right-line is that whoſe generating Point keeps always the ſame Direction: And a Curve-line is that whoſe generating Point always changes its Direction; the Side turning towards itſelf being called Concave, and that turning from itſelf Convex.

A Mathematical Curve is one whoſe Direction is always confined to the ſame fixed Law or Condition. And when that Law can be expreſſed by an Equation conſiſting of only one Varient, it is plain, that (in determining the Length of the Curve from that Equation) every Dimenſion of the Variant (Sect. VII.) muſt be reſtrained to the ſame Root. That is, in other Terms,

the

the Equation muſt be reduced to its Ultimate. For ſince, as has been ſhewn, every Equation conſiſts of as many Roots of its Variant, as that Variant conſiſts of Dimenſions; and ſuch Roots are generally of widely different Values: Hence, if the Equation be left, as it is in itſelf without any Reſtriction of its Roots, the Variant muſt have different Values; and conſequently the the Curve is not confined to the ſame Law, which is contrary to the Suppoſition. It follows therefore, that no Equation, of how many Variants ſoever it conſiſts, can expreſs the invariable Law of a Curve, but as every one of thoſe Variants under its different Dimenſions and Combinations with every other Variant is reſtrained to the ſame Root, or as it is reducible to an Ultimate. THEOREM I.

With Reſpect therefore to the Form of the Equations, by which the Law of any Curve is expreſſed, we ſhall uſe that known Diſtinction of Curves into *Tranſcendental*, and *Algebraical*.

A *Tranſcendental Curve* is one, which cannot be expreſſed by a fixed Algebraic Equation; as being not always fixed to the ſame Equation in every Point of the Curve.

An *Algebraical Curve* is ſuch a one, as in every Point expreſſed by the ſame Equation. And this again may be called either *Infinite*, when the Dimenſions of its Equation are infinite, and conſequently the Equation itſelf is expreſſed by an infinite Series; or *Finite*, when it is expreſſed in finite Terms.

And all thoſe *Finite Curves* are ſaid to belong to the ſame Family, whoſe Equations follow the ſame Form, *i. e.* have the ſame correſpondent variable and invariable Terms under the ſame Signs, with no other Difference than in the Dimenſions of thoſe Terms, by which they are denominated to be of a higher or lower *Order* in that *Family*. Thus, for Example, in the Circle, whoſe Diameter is $b = 2c$, Abſciſſa v, and Semiordinate s, as before; the Equation of the Circle of the firſt and *loweſt Order* (and indeed the only one, which can truly be called a Circle) is $bv - vv = ss$. The Equation of the Circle of the *ſecond Order* is $bv^2 - v^3 = s^3$, or $bbv - v^3 = s^3$: The Equation of the *third Order* is $bv^3 - v^4 = s^4$, or $b^2v^2 - v^4 = s^4$ or $b^3v - v^4 = s^4$: And univerſally the

Equation of Circles of *every Order* is $b^m v^n - v^{m+n} = s^{m+n}$. Where every one is at Liberty to ſubſtitute any poſſible Numbers for m and n, to which his Phancy ſhall guide him. And ſo for any other Curve. THEOREM II.

Now with Reſpect to their Generation or Deſcription, the prime Diviſion of Mathematical Curves is, I conceive, that they are all either ſuch as can have no two Parts of the ſame Curve in the ſame Plane; like the Worm of a Screw: Or elſe are ſuch as have all their Parts coincident with the ſame Plane. Of which latter Sort only, I ſhall treat here.

All Curves therefore capable of lying in the ſame Plane, with regard to their Generation, may be five Ways diſtinguiſhed. For,

Firſt. All ſuch Curves are either *Spirals*, which by ſome certain Law, keep continually revolving about the ſame Point or Center, (moveable or immoveable) till they terminate in that Point: Or *Vertical Curves* of which Sort are all the Curves in the foregoing Treatiſe (except the *Archimedean* and *Parabolic* Spirals in the eight Section), which on each Side the Vertex *V* include always equal and ſimilar Areas between the Curve itſelf, the Abſciſſa VP, and their reſpective right Semiordinates PM.

Secondly, Vertical Curves ere either *Aufugient*, which on each Side from the Vertex V, do always depart further from the Right Axis VP: Or *Revertent*, which, in their Departure on each Side from the Vertex V, do meet again in the ſame Point in the Axis.

Thirdly, Aufugient Curves are either *Determinate*, whoſe Right Axis is of a determinate Length, not to be exceeded; as the *Quadratrix* and *Cycloid*: Or *Indeterminate*, whoſe Right Axis together with the Curve on each Side, may be produced *ad infinitum*; as the *Parabola*, *Hyperbola*, &c.

Fourthly, Revertent Curves are either *Bipertite*, whoſe Wholes can be divided into no more than two Parts ſimilar and equal to each other, by the Right Axis; as the *Piriformis* and *Oval* to be ſeen hereafter: Or *Quadripertite*, whoſe Wholes are both divided into two ſuch congruous Parts by the Axis, and each of thoſe again into two other Parts ſimilar and equal to each other, by the Ordinate that biſects the Axis; as the *Circle* and *Ellipſis*.

Fifthly, Every *Algebraic Curve* univerſally may be ſaid to be *Conſtant*, when all thoſe of the ſame Sort and Order, expreſſed by the ſame Equation, do exactly keep the ſame Form, and are always perfectly ſimilar to each other; as the *Circle*, *Parabola*, *Ciſſoid*, *Cycloid*, &c. Or *Variable*, when thoſe of the ſame Sort and Order, expreſſed by the ſame Equation, may put on an Infinity of different Forms, at the utmoſt Diſſimilarity from each other; as the *Ellipſis*, *Hyperbola*, *Conchoid*, &c.

Here it will be eaſy to conceive that every Curve, whoſe Equation conſiſts of but one only Invariable Quantity, muſt be *Conſtant*. And on the contrary every Curve, whoſe Equation conſiſts of different Invariables, muſt be as *Variable* as thoſe included Invariables can have Differences between themſelves. THEOREM III.

Thus, if the Equation of a Family of Curves, wherein $b = 2d$, be $b^m v^n - v^{m+n} = \frac{d^{m+n}}{c^{m+n}} \times s^{m+n}$; then this Equation (as ſhall be ſhewn hereafter) may include in it a whole Family of both Ellipſes, and that of the Circles along with it. For the latter, as appears from their Equation above, is only a particular Example of this, wherein d or $\frac{1}{2}$ b and c are the ſame. *Conſectary* 1.

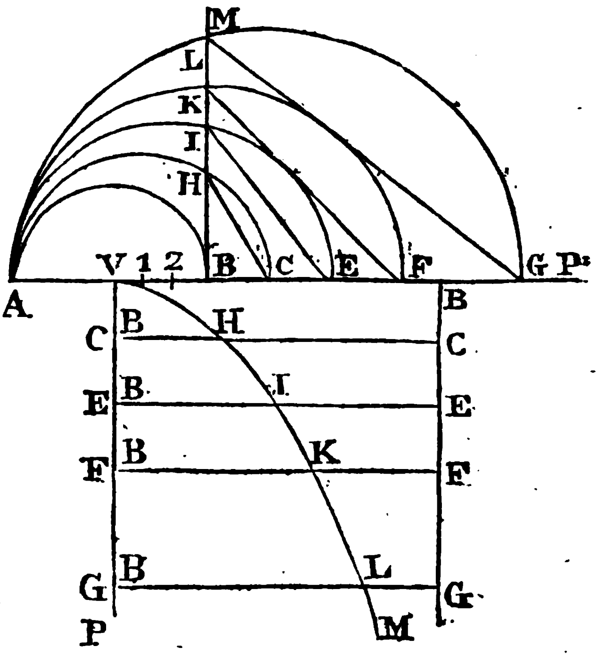
So again, if the Equation of a Family of Curves, wherein $b = 2d$, be $b^m v^n + v^{m+n} = \frac{d^{m+n}}{c^{m+n}} \times s^{m+n}$; then this is the Equation (as ſhall be ſhewn hereafter) of all both Oblique and Rectangular Hyperbolas. For the general Equation of the latter will be ſhewn to be $b^m v^n + v^{m+n} = s^{m+n}$, which is only a particular Example of the former, wherein after the ſame Manner d or $\frac{1}{2}$ b and c are the ſame. *Conſectary* 2.

Whence it appears, that each Order of Circles has exactly the ſame Relation to all the Variety of Ellipſes of the ſame Order; which the Rectangular Hyperbola of the like Order has to all its equipollent Oblique-angular Hyperbolas, whether Obtuſe or Acute. *Conſectary* 3.

II.

Of the Parabola.

SUPPOSE V to be the Center of a Circle, whoſe Diameter is $AB = b$, or whoſe Radius is $AV = VP = \frac{1}{2}b = p$, produced indefinitely from B to P; to which erect an indefinite Perpendicular from B to M. In the Right-line AVP &c. to the Right-hand of V, aſſume a Number of Points 1, 2, 3, 4, 5, &c. as ſo many Centers (the greater Number the better, eſpecially the nearer V), and with the Radii A 1, A 2, A 3, A 4, &c. deſcribe ſo many Semicircles; as AHC, AIE, AKF, ALG, &c. Draw VP, as an Axis perpendicular to AV, to which, at a convenient Diſtance draw BG parallel, and upon each of theſe from the Line AP lay off $BC = VC$, $BE = VE$, $BF = VF$, $BG = VG$, &c. as ſo many Abſciſſæ from V, drawing CC, EE, FF, GG,



all parallel to A P above. On each of which laſt Parallels again BH, BI, BK, BL, *&c.* of the reſpective Semi-circles above, as ſo many Semiordinates to their reſpective Abſciſſæ upon the Axis V P below. Then the Curve drawn through the Vertex V, and the Extremities H, I, K, L, *&c.* of theſe Semiordinates will be a Semi-parabola. For in every one of the Semicircles above, for Example the Semicircle A H C, becauſe A B is always equal to b or 2 p, if we put $BC = v$, and $BH = s$; then from the known Property of the Circle, we have $AB \times BC = \overline{BH}^2$ or $bv = ss$ or $2pv = ss$ the Equation of the *Parabola*, whoſe Right Parameter or *Latus rectum* is $AB = b = 2p$, *Conſect.* 4.

When therefore the Semiordinate s is the ſame with $p = VB$, we have $2pv = pp$. Whence $v = \frac{1}{2}p =$ the Diſtance upon the Axis V P of the Point called the *Focus* from the Vertex V. *Conſect.* 5.

Whence the univerſal Equation to the whole Family of *Parabolas* (Theo. 2.) muſt be $b^m v^n = s^{m+n}$. *Conſect.* 6.

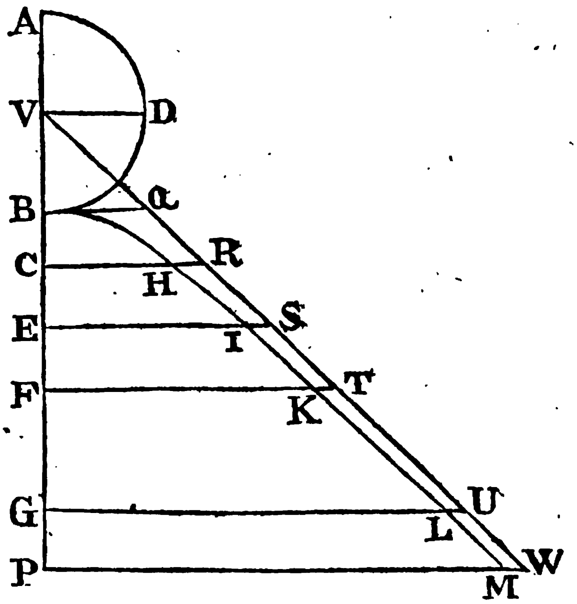
III.

Of the Rectangular Hyperbola.

IF in the foregoing Figure of Semicircles (to the ſame Parameter $AB = b$, and Abſciſſæ BC, BE, BF, BG, *&c.* $= v$) we make the reſpective Semiordinates equal to the Chords CH, EI, FK, GL, *&c.* $= s$; then we have (for Example) $\overline{BH}^2 + \overline{BC}^2 = \overline{CH}^2 = ss$, or $\overline{BH}^2 + vv = ss$. But by the laſt $\overline{BH}^2 = bv = 2pv$. Therefore the Equation of this Curve is $bv + vv = ss$ or $2pv + vv = ss$, the Equation of the *rectangular Hyperbola*, whoſe Parameter is $AB = b = 2p$. *Conſect.* 7.

When therefore the Semiordinate s is the ſame with $p = VD$, we have $2pv + vv = pp$. Whence $v = \overline{2^{\frac{1}{2}} - 1} \times p$ is the Focal Diſtance from the Vertex B. *Conſect.* 8.

And if we draw BQ parallel and equal to $VD = p$, producing the Hypothenuſe VQ of the Triangle VBQ to W: Then, ſince every Triangle VBQ, VCR, VES, VFT, *&c.* is right angled, and has its Legs $VB = BQ$, $VC = CR$, $VE = ES$, $VF = FT$, *&c.* it is plain we muſt every where have CR, ES, FT, GU, PW, *&c.* equal to $p + v$ or to $\overline{pp + ss}^{\frac{1}{2}}$. Conſequently every one of the laſt Lines muſt always be greater than its correſponding s, and yet muſt always the further the Curve is produced from its Vertex, approach nearer to an Equality with s, by the Portions HR, IS, KT, LU, MW, *&c.* which in the Value of their reſpective Semiordinates, will always be equal to $\overline{pp + ss}^{\frac{1}{2}} - s$. *Conſect.* 9.



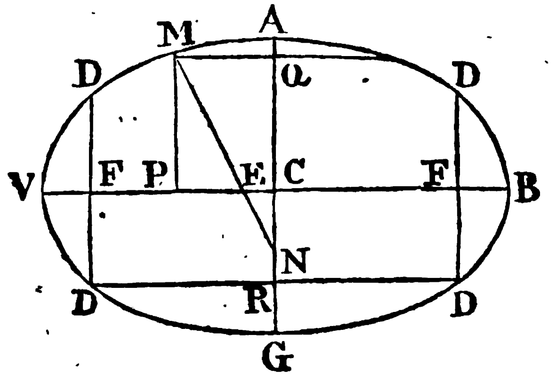
Whence the Right-line VW from the Property it thus receives (of approaching every where nearer and nearer to the Curve, without all Poſſibility of ever touching it) is called the *Aſymptote* of the Semi-curve.

The univerſal Equation therefore to the Family of *Rectangular Hyperbolas* (Theo. 2.) is $b^m v^n + v^{m+n} = s^{m+n}$. *Conſect.* 10.

IV.

Of the Ellipſis.

IF two Right-lines A G, V B of different Length biſect each other mutually at Right Angles in C; and a Ruler M N whoſe Length is equal to V C = C B = d be any how laid upon the ſame Plane; ſo as to have AC = CG = ME = c always above V B, and the lower Extremity any where in the Right-line C G. Then will the higher Extremity M be always in ſome Point of the Semi-curve VMAB. And the ſame being done on both Sides C G, with the Point N of the Ruler always in A G, and the Length M E = C G = c kept every where below C G; the lower Semi-curve V G B will be the ſame Way deſcribed by the Point M ſimilar and equal to the higher V M A B. The Property of which Curve will be had, as follows.



Draw the Semi-ordinate M P = s, whoſe Abſciſſa therefore muſt be V P = v. Whence PC = VC — VP = d — v. And VC — AC = MN — ME = EN = d — c. Then by Reaſon of the ſimilar Triangles E M P and E N C, we have compoundedly P E + E C : P E = M E + E N : M E or d — v : P E = d : c. Whence P E = $\frac{dc - cv}{d} = \overline{\overline{ME}^2 - \overline{MP}^2}^{\frac{1}{2}} = \overline{cc - ss}^{\frac{1}{2}}$, and $\overline{PE}^2 = \overline{c - \frac{c}{d}v}^2 = cc - ss$, or $cc - 2\frac{cc}{d}v + \frac{cc}{dd}vv = cc - ss$. Conſequently, the

Equation of this Curve is $2dv - vv = \frac{dd}{cc}ss$. The ſame with that of the *Ellipſis*, whoſe Tranſverſe Diameter is $b = 2d$, and its conjugate Diameter is $2c$. *Conſect.* 11.

Or, if we aſſume that Semiordinate or Right Parameter $DF = p$, between which and the Semitranſverſe $VC = CB = d$, the Semiconjugate $AC = CG = c$ is a mean Proportional; whence $cc = pd$, and $d = \frac{cc}{p}$: then ſubſtituting for cc or for d, the ſame Equation will be expreſſed by $2dv - vv = \frac{d}{p}ss$, or $\frac{2cc}{p}v - vv = \frac{cc}{pp}ss$. *Conſect.* 12.

Conſequently, if in the Equation $2dv - vv = \frac{d}{p}ss$, we make $p = DF = s$, we have $2dv - vv = dp$; whence $v = VF = BF = d - \overline{dd - dp}^{\frac{1}{2}} =$ the focal Diſtance upon the Tranſverſe VB. *Conſect.* 13.

Again, if we make $AQ = c - s = x$, and $MQ = d - v = \frac{cc}{d} - v = z$; then by ſubſtituting $c - x$ for s, and $\frac{cc}{p} - z$ for v in the Equation $\frac{2cc}{p}v - vv = \frac{cc}{pp}ss$, the Equation of the Curve becomes $\frac{p^2}{cc}z^2 = 2cx - xx$. Or putting $DR = FC = VF = \overline{dd - dp}^{\frac{1}{2}} = l = \overline{\frac{c^4}{p^2} - cc}^{\frac{1}{2}}$; whence $pp = \frac{c^4}{ll + cc}$; the ſame Equation ſtill is expreſſed by $2cx - xx = \frac{cc}{ll + cc}zz$. *Conſect.* 14.

Conſequently, if in the laſt Equation, we make $l = DR = z$, we have $2cx - xx = \frac{ccll}{cc+ll}$. Whence $x = RG = c - \frac{cc}{\overline{cc+ll}^{\frac{1}{2}}}$. equals the focal Diſtance upon the Conjugate AG. *Conſect.* 15.

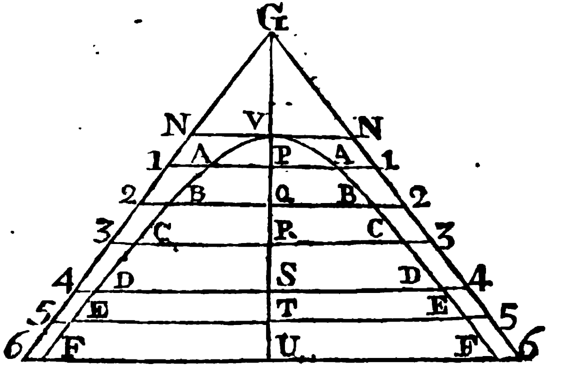
The univerſal Equation therefore to the Family of *Ellipſes* (*Conſect.* 11.) is $b^m v^n - v^{m+n} = \frac{d^{m+n}}{c^{m+n}} s^{m+n}$. *Conſect.* 16.

V.

Of the Hyperbola *in general.*

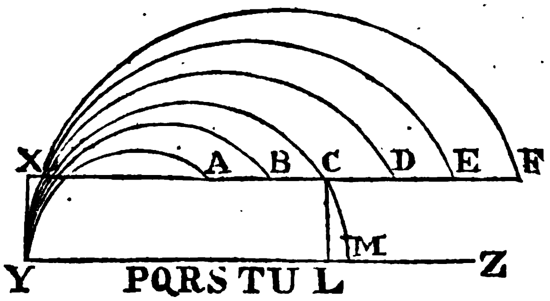
SUPPOSE a given rectilineal Iſoceles Triangle NGN, whoſe Baſe NN is biſected in V by the Perpendicular GV (*Fig.* 1.) and both Sides GN, GN, with the Perpendicular GV are indefinitely produced downwards to 6 and U. Let a Number of Lines be drawn below parallel to NN, meeting the produced Sides in 1, 2, 3, 4, 5, 6, &c. and cutting the Perpendicular in P, Q, R, S, T, U, &c. Make XY (Fig. 2.) equal to NV (Fig. 1.) to which at each Extremity erect the Perpendiculars XF, YZ; and with the Radii YP, YQ, YR, YS, YT, YU, *&c.* (Fig. 2.) which are reſpectively equal to P1, Q2, R3, S4, T5, U6, *&c.* (Fig. 1.) deſcribe the Arches YA, YB, YC, YD, YE, YF, *&c.* meeting XF in A, B, C, D, E, F, *&c.* Then lay the Extents XA, XB, XC, XD,

Fig. 1.



XD, XE, XF, *&c.* (Fig. 2.) reſpectively upon the Parallels (Fig. 1.) on both Extremities over G U. That is, from each Point 1 to its remoter Point A; ſo from 2 to B, 3 to C, 4 to D, 5 to E, 6 to F, *&c.* on each Side the ſame Way. And the Curve paſſing through the Points A, B, C, D, E, F, *&c.* on both Sides (Fig. 1.) from V will be an *Hyperbola*, whoſe Vertex is V, half tranſverſe Diameter is $GV = d$, half conjugate Diameter is $NV = XY = c$, N G N its Angle, and the indefinite Right-lines G6, G6, on each Side its Aſymptotes.

Fig. 2.



For the Equation of which Curve (for Example) we have (Fig. 1.) $GR = GV + VR = d + v$, and $CR = s$. And by Reaſon of the Similarity of the Triangles G N V and G 3 R, we have $GV : NV = GR : 3R$, or $d : c = d + v : \frac{cd + cv}{d} = 3R$. Therefore the Diſtance from 3 in the Aſymptote to its remoter Point C in the Curve muſt equal $\frac{cd+cv}{d} + s = \frac{cd+cv+ds}{d} = 3C + CR + RC$. Which is the ſame with $XC = YL$ (Fig. 2). And $\overline{LM}$ (Fig. 2.) muſt equal $YM - YL = 2YR - YL = \frac{2cd+2cv}{d} - \frac{cd+cv+ds}{d} = \frac{cd+cv-ds}{d}$. But from the known Property of the Circle (Fig. 2.) we have $YL \times LM = \overline{CL}^2 = \overline{XY}^2$. That is $\frac{cd+cv+ds}{d} \times \frac{cd+cv-ds}{d} = cc$. Whence $2dv + vv = \frac{dd}{cc}ss$ the Equation of the Hyperbola in general. Which, when G V is greater than N V, makes the Curve Acute-angular; when F V and N V are equal, Rectangular; and, when N V is greater than G V, Obtuſe-angular. *Confect.* 17.

And LM (Fig. 2.) $= \frac{cd + cv}{d} - s$ will always equal the Diſtance of the adjacent Aſymptote upon the Semiordinate produced. *Conſect*, 18.

Alſo, if (as in the Ellipſis) we aſſume that Semiordinate or Right Parameter equal p, between which and the Semi-tranſverſe d, the Semiconjugate c is a mean Proportional, whence $cc = pd$; then ſubſtituting for cc, the Equation will be $2dv + vv = \frac{d}{p} ss$, or (making $b = 2d$) $bv + vv = \frac{d}{p} ss$. Conſequently, when p equals s, the focal Diſtance in this Curve will be $v = \overline{dd + dp}^{\frac{1}{2}} - d$. *Conſect.* 19.

Therefore the univerſal Equation (Theo. 2.) to the Family of Hyperbolas is $b^m v^n + v^{m+n} = \frac{d^{m+n}}{c^{m+n}} s^{m+n}$. *Conſect.* 20.

VI.

Of the Piriformis.

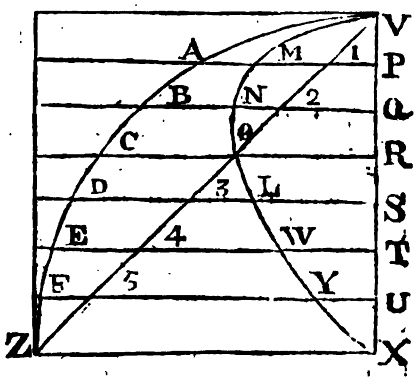
SUPPOSE the circular Quadrant VABCDEFZ, whoſe Center is X, Radii $VX = ZX$, and its Chord VZ. Let many Parallels be drawn to ZX from the Points P, Q, R, S, T, U, *&c.* to cut the Quadrant in A, B, C, D, E, F, *&c.* and the Chord in 1, 2, O, 3, 4, 5, *&c.* Then lay the Extent A1, B2, CO, D3, E4, F5, *&c.* reſpectively from P to M, Q to N, R to O, S to L, T to W, U to Y, *&c.* And the Curve-line paſſing through the Points V, M, N, O, L, W, Y, *&c.* X, with the like on the other Side of of the Axis VX will form a Curve in the Reſemblance of a Pear. Which for that Reaſon may be called a *Piriformis.* Whoſe Abſciſſæ are VP, VQ, VR, *&c.* and their reſpective Semiordinates PM, QN, RO, *&c.* it is re-

required to determine the Law, *&c.* of this Curve. Suppoſing the given Radius $VX=ZX=c$, the Semiordinate $MP=s$, and its reſpective Abſciſſa $VP=v$.

Here we know, from the Property of a Rightangled Triangle, that $\overline{VX}^2 + \overline{ZX}^2 = 2cc = \overline{VZ}^2$, or $VZ = \overline{2cc}^{\frac{1}{2}} = 2^{\frac{1}{2}} \times c$. And from the known Property of the Circle $\overline{AP}^2 = \overline{2VX - VP} \times VP$, or $AP = \overline{2cv - vv}^{\frac{1}{2}}$. But $VP = v = 1P$, and from the Deſcription of the Curve $A1 = MP = AP - 1P = AP - VP = \overline{2cv - vv}^{\frac{1}{2}} - v = s$, the Eqnation of the Curve. *Conſect.* 21.

Therefore, to make s the greateſt, we have $\frac{cv^{\circ} - vv^{\circ}}{\overline{2cv - vv}^{\frac{1}{2}}} - v^{\circ} = 0$. Whence in that Caſe, $v = c \times \overline{1 - \overline{0.5}^{\frac{1}{2}}} = VX - \frac{1}{2} VZ$, and $s = c \times \overline{2^{\frac{1}{2}} - 1} = VZ - VX$. *Conſect.* 22.

Alſo putting $b = 2c$, whence the Equation becomes $\overline{bv - vv}^{\frac{1}{2}} - v = s$, we have sv° (Sect. IX. Theo. 52.) $= \overline{bv - vv}^{\frac{1}{2}} v^{\circ} - vv^{\circ} =$ the Ultimator of every Area MVPM, But $\overline{bv - vv}^{\frac{1}{2}} v^{\circ}$ (Sect. IX. *Example* 7.) is the Ultimator of the correſpondent circular Area AVPA, and vv° is the Ultimator of $\frac{1}{2} vv$ or of half the Square of VP: Therefore in this Curve every Area MVPM is equal to the Difference between its half correſpondent circular Segment AVPA and half the Square of the Abſciſſa VP. Conſequently the Area AVMA is always equal to half the Square of the Abſciſſa VP. *Conſect.* 23, 24.

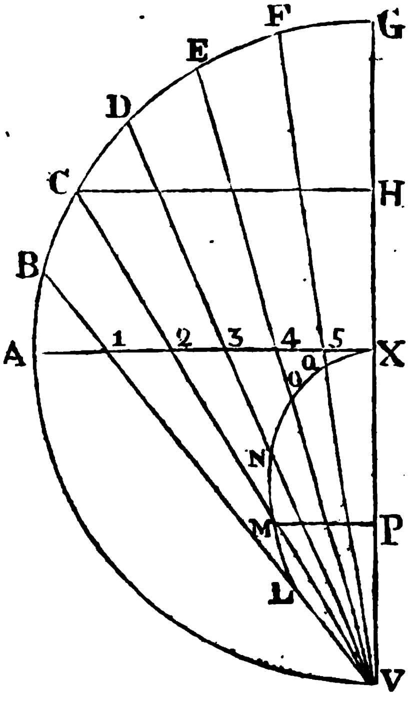


Hence it follows, that the Area of the whole *Piriformis* is equal to the Difference between the Area of its circumſcribing Semi-circle, and the Square of its Semidiameter or Radius *Conſect.* 25.

Laſtly, The univerſal Equation (Theo. 2.) to the Family of Piriforms is $\overline{b^m v^n - v^{m+n}}|^{\frac{1}{m+n}} - v = s$. *Conſect.* 26.

VII.

Of the Oval.



SUPPOSE a Circle, whoſe Center is X, upper Quadrant AGX, and lower Quadrant AVX, upon the Diameter GV $= b = 2c$. Let any Number of Chord-lines BV, CV, DV, *&c.* be drawn from the upper Quadrantal Arch AG to the lower Extremity V of the Diameter GV, cutting AX in 1, 2, 3, *&c.* and lay the 1 V, 2 V, 3 V, &c. reſpectively from B to L, C to M, D to N, *&c.* then the Curve-line paſſing through the Points V, L, M, N, O, Q, *&c.* on both Sides the Axis V X from its Reſemblance to an Egg may be called an *Oval*; the Proportion between whoſe Right Semiordinates and their reſpective Abſciſſæ is required to be determined.

Let (for Example) the Semiordinate be $MP = s$, whoſe Abſciſſa is $VP = v$. Whence $VM = \overline{vv+ss}^{\frac{1}{2}}$, and $VP : VM = VX : V2$, or $v : \overline{vv+ss}^{\frac{1}{2}} = c : \frac{c}{v} \times \overline{vv+ss}^{\frac{1}{2}} = V2 = CM$

by Conſtruction. Then $CM + VM = VC = \frac{c+v}{v}\times\overline{vv+ss}^{\frac{1}{2}}$; and $VM : VP = VC : VH$, or $\overline{vv+ss}^{\frac{1}{2}} : v = \frac{c+v}{v}\times\overline{vv+ss}^{\frac{1}{2}} : c+v = VH$. Whence $HX = v$, and $GH = c-v$; and from the known Property of a Circle (*viz.* $GH\times VH = \overline{CH}^{2} = \overline{c-v}\times\overline{c+v}$) we have $\overline{cc-vv}^{\frac{1}{2}} = CH$. But again, from the Similarity of the Triangles, we have $VP : MP = VH : CH$, or $v : s = c+v : s\times\frac{c+v}{v} = CH$. Therefore $s\times\frac{c+v}{v} = \overline{cc-vv}^{\frac{1}{2}}$, or $ss\times\frac{cc+2cv+vv}{vv} = cc-vv$, or (dividing both Sides by $c+v$) $ss\times\frac{c+v}{vv} = c-v$. Whence the Equation of this Curve is $ss = \frac{cv^{2}-v^{3}}{c+v}$. *Conſect.* 27.

Therefore to make s here the greateſt, that it poſſibly can be, we have $\frac{2c^{2}vv^{\circ}-2cv^{2}v^{\circ}-2v^{3}v^{\circ}}{cc+2cv+vv} = 0$. Whence (dividing by $\frac{2vv^{\circ}}{\overline{c+v}^{2}}$) we have $cc-cv-vv=0$, or $v = \frac{1}{2}c\times\sqrt{5^{\frac{1}{2}}-1}$.

Whence $s = c\times\frac{\sqrt{7-3\times5^{\frac{1}{2}}}^{\,\frac{1}{2}}}{\overline{1+5^{\frac{1}{2}}}^{\frac{1}{2}}}$. *Conſect.* 28.

And if we put $XP = u$ (taking the Abſciſſa from X); then ſubſtituting $c-u$ for v in the 27th Conſectary, we have $ss = \frac{c^{2}u-2cu^{2}+u^{3}}{2c-u}$ for the Equation of the Curve. *Conſect.* 29.

According to which laſt, the univerſal Equation of the Family of ſuch Ovals will be $b^n s^n - u^n s^n = \overline{\tfrac{1}{2} b - u}^{\,n} \times u^n$. *Conſectary* 30.

VIII.

Of the Ciſſoid.

LET XE be a Tangent to the Diameter $VX = b$ of the Semicircle VNLOX; ſee the Figure to *Example* 18. Sect. IX. To which Tangent XE on every one of its Secants VB drawn from the other Extremity V of the Diameter make VM equal to OB; then the Curve line drawn from V through every one of thoſe Points M is by *Diocles* the Inventor called a *Ciſſoid*, whoſe Abſciſſæ are $VP = ZM = v$, and their reſpective Semiordinates $PM = VZ = s$. Conſequently $ZX = MA = b - s$, ſince ZL is parallel to VP, and both parallel to XE.

Draw OR parallel to VX, and OS parallel to XE. Then becauſe by Conſtruction, we have OB equal to VM, and the Triangles OBR and VMP are ſimilar; therefore $PM = OR = SX = s$; whence $VS = b - s$. Conſequently, from the Nature of the Circle $VS \times SX = bs - ss = \overline{SO}^2$. But again from the Similiarity of the Right angled Triangles VZM and VSO we have $VZ : ZM = VS : SO$, or $s : v = b - s : v \times \frac{b-s}{s} = SO$; whence $vv \times \frac{bb - 2bs + ss}{ss} = \overline{SO}^2 = bs - ss$. And dividing both Sides by $b - s$ we have $vv \times \frac{b-s}{ss} = s$, or $vv = \frac{sss}{b-s}$ the Equation of the Curve. *Conſect.* 31.

Hence, becauſe here $ZX = VS = b - s$, whence $ZL = SO = v \times \frac{b-s}{s}$. Therefore in the *Ciſſoid* of the Circle we have always $VZ : ZX = ZM : ZL$. *Conſect.* 32.

It appears then (ſee the ſame Figure as in the laſt) that every *Revertent* Curve VNLOX may have its peculiar Ciſſoid VMQ, by making VM ſtill the ſame Way equal to OB; the Right Axis of the generating *Revertent* being VX, to which VT and XE are perpendicular at each Extremity. Whence, if VT be the Axis of the generated Ciſſoid; then XE will be its Aſymptote, and the Semi-ordinate $PM = VZ$ of the Ciſſoid will always equal OR or SX, the Complement of the correſpondent Abſciſſa VS (of the *Revertent*) to its Axis VX. That is $PM = SX = VX - VS$. *Conſect.* 33.

Alſo in every Ciſſoid VMQ, As the Semiordinate $PM = VZ$ is to its Abſciſſa $VP = ZM$, ſo is the correſpondent Abſciſſa VS of the generating *Revertent* to its Semiordinate SO. *Conſect.* 34.

For *Example*, if the generating *Revertent* be a *Piriformis*, where $VX = c$, the Extremity neareſt its greateſt Semiordinate being V; then putting $VP = ZM = v$, and $PM = VZ = SX$ (*Conſect.* 33.) $c = s$, we have $VS = VX - SX = c - s$. Which laſt (being ſubſtituted for v, *Conſect.* 21.) gives $\overline{cc - ss}^{\frac{1}{2}} - c + s = SO$. But (by *Conſect.* 34.) $v \times \frac{c-s}{s} = SO$. Whence $\overline{cc - ss}^{\frac{1}{2}} - c + s = v \times \frac{c-s}{s}$, or $\frac{s \times \overline{cc - ss}^{\frac{1}{2}}}{c-s} - s = v$, or $\left.\frac{c+s}{c-s}\right|^{\frac{1}{2}} s - s = v$, the Equation of this *Ciſſoid*. *Conſect.* 35.

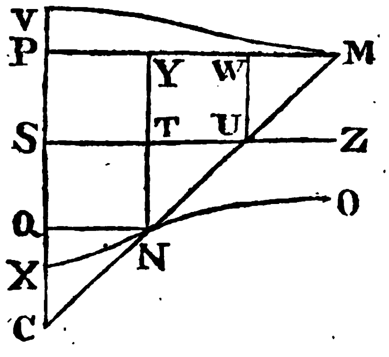
Again, if the generating *Revertent* be an Oval, whoſe Axis $VX = c$, the Extremity neareſt its greateſt Ordinate being V; then putting $VP = ZM = v$, and $PM = VZ = SX$ (*Conſect.* 33.) $= s$, we have $VS = VX - SX = c - s$. Which laſt (being ſubſtituted for u, *Conſect.* 29.) gives $\frac{cs^2 - s^3}{c+s} = \overline{SO}^2$. But (*Con-*

ſect. 34.) $v \times \frac{c - s}{s} = SO$, or $v^2 \times \frac{cc - 2cs + ss}{ss} = \overline{SO}^2$. Whence $v^2 \times \frac{cc - 2cs + ss}{ss} = \frac{cs^2 - s^3}{c + s}$, or (dividing by $c - s$) $v^2 \times \frac{c - s}{ss} = \frac{ss}{c + s}$). Conſequently $vv = \frac{ssss}{cc - ss}$ is the Equation of this *Ciſſoid*. *Conſect.* 36.

IX.

Of the Conchoid.

SUPPOSE two Right-lines CV, SZ interſecting each other at Right-angles in S, and from the Extremity C of the former, a Number of Right-lines CM &c. be drawn on both Sides CV cutting SZ in S, U, &c. Then if equal Extents be laid upon all thoſe Lines drawn thus from the Point C on both Sides SZ, *viz.* $VS = SX = MU = UN =$ &c. the Curve-lines (drawn through the Points V, M, &c. above, and through X, N, &c. below) are by their Inventor *Nicomedes* called a *Firſt* and *Second Conchoid.* To both which C is the Center; SW the Aſymptote; $VS = SX = MU = UN =$ &c. $= p$ their Radius; and $CS = b$ their Diameter. Whence, putting $PS = SQ = YT = TN =$ &c $= a = UW$, we have $CP = CS + PS = b + a$, and $CQ = CS - SQ = b - a$. Alſo, by Reaſon of the ſimilar Triangles UWM and CSU, we have $UW : MU = CS : CU$, or $a : p = b : \frac{bp}{a} = CU$.



In the *Firſt Conchoid* therefore, we have $CU + MU = \frac{bp}{a} + p = p \times \frac{b+a}{a} = CM$. Whence, putting the Semiordinate $PM = s$, we have $\overline{CP}^2 + \overline{PM}^2 = \overline{CM}^2$, or $\overline{b+a}^2 + s^2 = \frac{pp}{aa} \times \overline{b+a}^2$, or $s^2 = \frac{pp-aa}{aa} \times \overline{b+a}^2$ the Equation of the Curve. *Conſect.* 37.

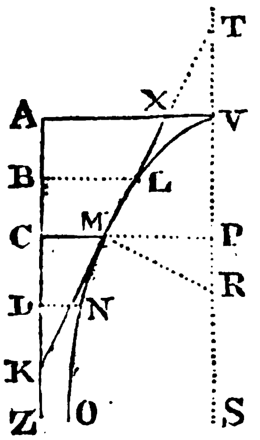
And in the *Second Conchoid*, we have $CU - NU = \frac{bp}{a} - p = p \times \frac{b-a}{a} = CN$. Whence, putting the Semiordinate $QN = z$, we have $\overline{CQ}^2 + \overline{QN}^2 = \overline{CN}^2$, or $\overline{b-a}^2 + z^2 = \frac{pp}{aa} \times \overline{b-a}^2$, or $z^2 = \frac{pp-aa}{aa} \times \overline{b-a}^2$ the Equation of the Curve. *Conſect.* 38.

X.

Of the Logarithmic Curve.

SUPPOSE a Right-line AZ be divided into any Number of equal Parts (as A = BC = CD = &c.) and at the Points of Diviſion, a Series of Perpendiculars are raiſed in continued Proportion (that is AV : BL = BL : CM = CM : DN = &c.); the ſame continued Proportion between the Perpendiculars, being ſtill obſerved, when each of theſe equal Parts AB, BC, CD, &c. is divided into ever ſo many other equal Parts; then the Line drawn through the other Extremities V, L, M, N, &c. of all theſe Perpendiculars, will conſtitute what is called a *Logarithmic Curve*, whoſe *Abſciſſæ* AB, AC, AD, &c. are the Logarithms of their reſpective Ordinates BL, CM, DN, &c. And AZ &c. is an Aſymptote. *Conſect.* 39.

Hence, if B L = r × A V; then will C M = r^2 × A V, D M = r^3 × A V, &c. Whence univerſally (making A V equal to Unity) if A C = n × A B, we ſhall have C M = r^n × B L. *Conſect.* 40.



Therefore this Curve is of the Sort, which we have called *Variable*; its Varieties being as infinite as the whole Numbers by which the Exponent n may be expreſſed, ſuppoſing the Letter n to be a Fraction; and again in each of theſe the different Ratios between the Abſciſſa and Semiordinate is alſo infinite. *Conſect.* 41.

If we ſuppoſe the Curve to be ſuch, as has A V equal to Unity, and A B, A C, A D, &c. expreſſing the Logarithm of every Number below Unity, beginning from A=0 for the Logarithm of Unity; then draw VS parallel to AZ for the Axis of the convex Curve VL. MNO, producing any of the foregoing Ordinates to the Axis (for Example C M to P) and put P M = s; whence C M = 1 — s, whoſe known natural Logarithm is $-s-\frac{1}{2}s^2-\frac{1}{3}s^3-\frac{1}{4}s^4-\frac{1}{5}s^5-\frac{1}{6}s^6-$ &c. = — A C = — V P = — v. The Ultimator therefore of V P is $-s^0-ss^0-s^2s^0-s^3s^0-s^4s^0-s^5s^0-$ &c. $=-\frac{s^0}{1-s}$. *Conſect.* 42.

Under which Suppoſition, if it be required to draw a Tangent through the Point M, we have this Proportion (Sect VIII.) as the Ultimator s^0 of the Semiordinate M P is to the Ultimator $\frac{s^0}{1-s}$ of the Abſciſſa V P, ſo is the Semiordinate s = M P to the Subtangent $\frac{s}{1-s}$ = T P. But from the ſimilar Triangles P M T and C M K, we have M P : T P = C M : C K, or $s : \frac{s}{1-s} = 1-s : 1$ = the oppoſite Subtangent C K. Therefore C K in this Curve muſt be an invariable Quantity. *Conſect.* 43.

Next draw the Normal M R perpendicular to the Tangent T M K. Then $\overline{CK}^2+\overline{CM}^2$ or $2-2s+ss=\overline{MK}^2$; whence $MK=\overline{2-2s+ss}^{\frac{1}{2}}$. And $CK:MK=MP:MR$, or $1:\overline{2-2s+ss}^{\frac{1}{2}}=s:s\times\overline{2-2s+ss}^{\frac{1}{2}}=$ the Normal M R. *Conſect.* 44.

Therefore for finding the Length of the Arch V L M, as in all other Curves, ſay, As the Semiordinate $MP=s$ is to the Normal $MR=s\times\overline{2-2s+ss}^{\frac{1}{2}}$, ſo is the Ultimator $\frac{s^0}{1-s}$ of the Abſciſſa V P to the Ultimator $\frac{s^0\times\overline{2-2s+ss}^{\frac{1}{2}}}{1-s}$ of the Arch VLM. Then putting $cc=2$, the Ultimator of the ſaid Arch may be expreſſed by $\frac{s^0\times\sqrt{\overline{cc-2s+\frac{2ss}{cc}}}^{\frac{1}{2}}}{\frac{1}{2}cc-ss}=c\,s^0$ multiplying the Series $\frac{2}{cc}+\frac{2s}{c^4}+\frac{5s^2}{c^6}+\frac{11s^3}{c^8}+\frac{91s^4}{4c^{10}}+\frac{183s^5}{4c^{12}}+\frac{729s^6}{8c^{14}}+\frac{2909s^7}{8c^{16}}+$ &c. Subject is c multiplying the Series $s+\frac{s^2}{c^4}+\frac{5s^3}{3c^6}+\frac{11s^4}{4c^8}+\frac{91s^5}{20c^{10}}+\frac{183s^6}{24c^{12}}+\frac{729s^7}{56c^{14}}+\frac{2909s^8}{64c^{16}}+$ &c. equal to the Length of the Arch V L M. *Conſect.* 45.

To find the Area VLMPV, for the Ultimator $s\,v^0$ (Sect. IX. Theo. 52.) we have $s\times\frac{s^0}{2-1}=s\,s^0+s^2\,s^0+s^3\,s^0+s^4\,s^0+s^5\,s^0+$ &c. whoſe Subject $\frac{1}{2}s^2+\frac{1}{3}s^3+\frac{1}{4}s^4+\frac{1}{5}s^5+\frac{1}{6}s^6+$ &c. is equal to the ſaid Area. *Conſect.* 46.

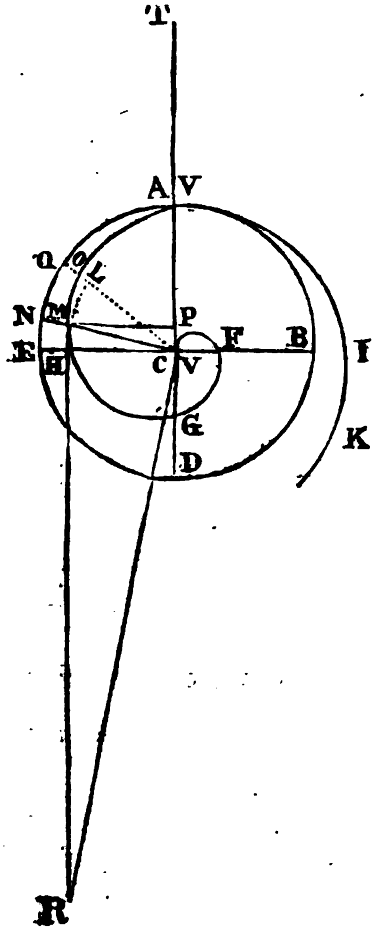
Whence, the Learner may perceive how to proceed with this Curve, as with the Hyperbola and Ciſſoid in the laſt Section; obſerving that A V is a fixed Quantity.

Of

XI.

Of Spirals *in general.*

LET the Surface of the Paper here be conſidered as a Plane of an indefinite Extent, upon which C is aſſumed as a fixed Center, and around that indefinite the Radius C T is conceived to be continually moved in the Direction from A to B, D, E, N, A, B, D, &c. with a Velocity conſtantly the ſame; ſuppoſing that at the ſame Time, when C T begins that Motion, a moveable Point V coincident at firſt with the Center, C begins to proceed with another different uniform Velocity along the ſame moveable Radius CT, in ſuch Sort, that when C T coincides with C B, the Point V is in F, when CT coincides with CD, the Point V is in G, when C T coincides with C E, the Point V is in H; and ſo on till V paſſes through M, O, A, I, K, &c. Then the Curve C F G H M O A I K &c. which is thus formed, is called a *Spiral.* To which every Circle deſcribed upon the ſame Plane on the Center C may be ſaid to be *Terminant*, with this Diſtinction; that every ſuch Circle, as A B D E A here (which paſſes through the Point V, when the moveable or generating Radius C T has come to its firſt Poſition, or made an entire Revolution) is called *Conterminant*, and every other terminant Circle, *Diſterminant*. And with reſpect to the Number of entire Revolutions



thus made by the generating Radius, the whole Spiral is ſaid to be *Firſt*, *Second*, *Third*, &c.

In all Spirals therefore, if we put c always for the Number of the Revolutions, and r = the Radius of any conterminant Circle, whoſe Circumference is q, we ſhall have c q equal to the Circumference of every other conterminant Circle. Whence, if r be the Radius which bounds the firſt Spiral, as A C here; the Circumference of the firſt Conterminant will be c q = q, of the ſecond c q = 2 q, of the third c q = 3 q, &c. And the whole Spiral included, will be the *Firſt*, *Second*, *Third*, &c. accordingly. And in every Diſterminant Circle, the Letter c muſt be a mixed Number or Fraction. THEOREM IV.

If we put AC = r, the common Radius of any terminant Circle with its included Spiral, and N A = u an Arch of that Circle bounded by the Radius N C = r, whoſe correſpondent Spiral Radius is MC = y, and s = the Right Sine of that Arch: Then becauſe r : s = y : $\frac{sy}{r}$, we have M P = $\frac{sy}{r}$ for the Correſpondent Ordinate of the Spiral; and for the ſame Reaſon $y\times\overline{rr-ss}^{\frac{1}{2}} = CP$. But from the Menſuration of the Arch of a Circle, we have $s = u - \frac{u^3}{2\times3r} + \frac{u^5}{2\times3\times4\times5r^6} - \frac{u^7}{2\times3\times4\times5\times6\times7r^6} +$ &c. Therefore $MP = \frac{sy}{r} = y$ multiplying the Series $\frac{u}{r} - \frac{u^3}{2\times3r^3} + \frac{u^5}{2\times3\times4\times5r^5} - \frac{u^7}{2\times3\times4\times5\times6\times7r^7} +$ &c. THEOREM V.

In every terminant Spiral therefore, when u is equal to E A the Quadrant $\left(\frac{q}{4}\right)$ of the Circle, then the Ordinate $\frac{sy}{r}$ and the Spiral Radius H C = y are the ſame, becauſe in that Caſe s = r. And when the Circle and Spiral are conterminant we have always four ſuch; *viz. firſt* A O M H, whoſe greateſt Spiral Radius is A C and leaſt H C; *ſecondly*, H G whoſe greateſt Spiral Radius is H C and leaſt C G; *thirdly*, G F, whoſe greateſt Spiral Radius is C G and leaſt C F; *fourthly*, F C, whoſe greateſt Spiral Radius is C F and the leaſt is equal to nothing in the firſt Spiral, or to the greateſt Radius of the firſt Spiral in the ſecond, of the ſecond in the third, of the third in the fourth, &c. THEOREM VI.

Imagine the two Radii N C and Q C to be infinitely near one another, and through the Point M where the former cuts the Spiral deſcribe the circular Arch M L parallel to the terminate Circle, which makes the Triangle M L O right angled at L; whence we have $NQ = u^{\circ}$ and $OL = y^{\circ}$, and $NC : MC = NQ : ML$, or $r : y = u^{\circ} : \frac{yu^{\circ}}{r} = ML$. Conſequently, by the common Principles of Geometry, the Area of the Triangle M O C is equal to $\frac{1}{2} OC \times ML = \frac{yyu^{\circ}}{2r}$ the Ultimator of the Spiral Space MCAOM. THEOREM VII.

Suppoſe R M a Tangent to the Point M in the Curve, whoſe Subtangent R C is perpendicular to the Spiral Radius $MC = y$ in the Center C. Then putting $RC = d$, and $RM = p$, we have $\overline{RM}^2 = \overline{RC}^2 + \overline{MC}^2$, or $p = \overline{dd + yy}^{\frac{1}{2}}$. THEOREM VIII.

Hence the two Triangles R C M and M L O are ſimilar. For under this Suppoſition, the Curvilineal Particle M O muſt coincide with the Tangent R M produced, and the circular Arch M L muſt be parallel to the Subtangent R C. Conſequently, $OL : ML = MC : RC$, or $y^{\circ} : \frac{yu^{\circ}}{r} = y : \frac{yyu^{\circ}}{ry^{\circ}} = RC =$ d the Subtangent. THEOREM IX.

Hence laſtly, $RC : RM = ML : MO$, or $d : p = \frac{yu^{\circ}}{r} : \frac{pyu^{\circ}}{dr}$ the Ultimator of the Spiral Arch M O A. THEOREM X.

This laſt Theorem is the ſame with what is commonly uſed for determining the Lengths of all Spiral Lines; but ſcarce any Dependence is to be placed upon it: For all the particular Examples, which I have ſeen drawn from it, ſeem only as ſo many

Inſtances of the exceeding Vanity of ſome Perſons, who appear reſolved always to ſet down ſomething, rather than once appear ignorant in any Thing. Of which ſee more in the Concluſion.

XII.

Of the Direct Spiral.

THE *Direct* Spiral, or (as it is otherwiſe called from its great Inventor *Archimedes*) the *Archimedean* Spiral, is, when in its Deſcription the Point V moves always the ſame Part of the Radius $AC = r$, which the Point A does of the whole Revolution cq. That is (putting $a = cq - u$, whence $a^\circ = - u^\circ$) $y : r = a : cq$. Conſequently in this Spiral we have $y = \frac{ra}{cq} = r - \frac{ru}{cq}$, $a = \frac{cqy}{r}$, and $u = cq - \frac{cqy}{r}$. *Conſect.* 47.

Hence in the firſt Spiral ($c = 1$), when $u = \frac{1}{4}q$, we have $HC = y = r - \frac{ru}{q} = \frac{3}{4}r$; when $u = \frac{1}{2}q$, we have $GC = y = \frac{1}{2}q$; when $u = \frac{3}{4}q$, we have $CF = y = \frac{1}{4}q$. *Conſect.* 48.

Then for the Spiral Area MCAM (Theo. 7.) for $\frac{yyu^\circ}{2r}$ or $\frac{yya^\circ}{2r}$ $\left(a^\circ = \frac{cqy^\circ}{r}\right)$ we have $\frac{cqy^2y^\circ}{2rr}$ its Ultimator, whoſe Subject $\frac{cqy^3}{6rr}$ gives the Complement of the Area MCAM. Conſequently, when y is the ſame with r, the Whole is equal to $\frac{cqr}{2} \times \frac{1}{3}$. Otherwiſe, ſince $yy = r^2 \times \frac{c^2q^2 - 2cqu + u^2}{ccqq}$, for $\frac{yyu^\circ}{2r}$ we have $ru^9 \times \frac{c^2q^2 - 2cqu + uu}{2ccqq}$: Whoſe Subject $\frac{1}{2}ru - \frac{ruu}{2cq} + \frac{ruuu}{6ccqq}$ gives the Area MCAM itſelf. Which when u equals q (as it can never be equal to more for the Area within the ſame Circle) gives

$\frac{rq}{2} \times \frac{3cc - 3c + 1}{3cc}$ for the Area of the whole Spiral Space AMHGFCA. So that the firſt Spiral Area is $\frac{rq}{2} \times \frac{1}{3}$, the ſecond $\frac{rq}{2} \times \frac{7}{12}$, the third $\frac{rq}{2} \times \frac{19}{27}$, the fourth $\frac{rq}{2} \times \frac{37}{48}$, the fifth $\frac{rq}{2} \times \frac{61}{75}$, &c. Where r and q are always the Radius and Circumference of the greateſt Circle. *Conſect.* 49.

Again, becauſe $y^{\circ} = \frac{ra^{\circ}}{cq} = -\frac{ru^{\circ}}{cq}$, $a^{\circ} = \frac{cqy^{\circ}}{r}$, and $u^{\circ} = -\frac{cqy^{\circ}}{r}$; therefore (Theo. 9.) for $\frac{yyu^{\circ}}{ry^{\circ}}$ or $\frac{yya^{\circ}}{ry^{\circ}}$ we have $\frac{aa}{cq}$ or $\frac{cqyy}{rr} = d =$ the Subtangent R C. *Conſect.* 50.

Whence, the Subtangent at the Point A (when $a = cq$, and $y = r$) will always equal cq. And (Conſect. 48.) if $c = 1$, the Subtangent at H ($HC = y = \frac{3}{4}r$, $a = \frac{3}{4}$) is $\frac{9}{16}q$; the Subtangent at G ($GC = y = \frac{1}{2}r$, $a = \frac{1}{2}$) is $\frac{1}{4}q$; the Subtangent at F ($FC = y = \frac{1}{4}r$, $a = \frac{1}{4}$) is $\frac{1}{16}q$. *Conſect.* 51.

Laſtly, in the firſt Spiral, when $c = 1$, we have $RC = d$ (Conſect. 50.) $= \frac{qq - 2qu + uu}{q}$, and $y = r \times \frac{q-u}{q}$ whence (Th. 8.) $RM = p = \frac{q-u}{q} \times \overline{rr + qq - 2qu + uu}^{\frac{1}{2}}$. Therefore (Theo. 10.) for $\frac{pyu^{\circ}}{dr}$ we have $\frac{u^{\circ}}{q} \times \overline{rr + qq - 2qu + uu}^{\frac{1}{2}}$ or (putting $1 = r^2 + q^2$) $\frac{u^{\circ}}{q} \times \overline{1 - 2qu + uu}^{\frac{1}{2}} = \frac{u^{\circ}}{q} - u u^{\circ} + \frac{rr}{2q} u^2 u^{\circ} + \frac{r^2}{2} u^3 u^{\circ} + \frac{4q^2r^2 - r^4}{8q} u^4 u^{\circ} + \frac{4q^2r^2 - 3r^4}{8} u^5 u^{\circ} + \frac{8q^4r^2 - 12q^2r^4 + r^6}{16q} u^6 u^{\circ} +$ &c. Whoſe Subject therefore, when u becomes equal to q, will be $1 - \frac{1}{2}q^2 + \frac{1}{6}r^2q^2 + \frac{1}{8}r^2q^4 + \frac{4q^2r^2 - r^4}{40} q^4 + \frac{4q^2r^2 - 3r^2}{48} q^6 + \frac{8q^4r^2 - 12q^2r^4 + r^6}{112} q^6 +$ *&c.* for the Length of the whole firſt Spiral. *Conſect.* 52.

Otherwiſe, becauſe $\overline{RM}^2 = \overline{RC}^2 + \overline{MC}^2$, or $\overline{\frac{a^4}{c^2q^2}+yy}^{\frac{1}{2}} = p$, and $y = \frac{ra}{cq}$; therefore in the firſt Spiral (Theo. 10.) for $\frac{pya^{\circ}}{dr}$ we have $\frac{a^{\circ}}{q} \times \overline{a^2+rr}^{\frac{1}{2}}$, or (by making $r=1$) $\frac{a^{\circ}}{q} \times \overline{aa+1}^{\frac{1}{2}} = \frac{aa^{\circ}}{q} + \frac{a^{\circ}}{2qa} - \frac{a^{\circ}}{8qa^3} + \frac{a^{\circ}}{16qa^5} - \frac{5a^{\circ}}{128qa^7} + \frac{7a^{\circ}}{256qa^9} - \frac{21a^{\circ}}{1024qa^{11}} +$ &c. But the Subject of the ſecond Term $\frac{a^{\circ}}{2qa}$ (Sect. VII. Theo. 20, 24.) is 0, therefore the Subject of the whole Series becomes $\frac{aa}{2q} + * + \frac{1}{16qa^2} - \frac{1}{64qa^4} + \frac{5}{768qa^6} - \frac{7}{2048qa^8} + \frac{21}{10240qa^{10}} -$ &c. And, if for a we put its Complement to the Circle, whence $\frac{a^{\circ}}{2qa}$ becomes $\frac{1}{2q} \times \frac{a^{\circ}}{q-a} = \frac{a^{\circ}}{2q}$ multiplying the Series $\frac{1}{q} + \frac{a}{q^2} + \frac{a^2}{q^3} + \frac{a^3}{q^4} + \frac{a^4}{q^5} +$ &c. we have the Subject $\frac{a}{2q^2} + \frac{a^2}{4q^3} + \frac{a^3}{6q^4} + \frac{a^4}{8q^5} + \frac{a^5}{10q^6} +$ &c. to be added to the former. So that, according to this, when a equals q, we have $\frac{1}{2}q + \frac{1}{16q^3} - \frac{1}{64q^5} + \frac{5}{768q^7} - \frac{7}{2048q^9} + \frac{21}{10240q^{11}} -$ &c. added to $\frac{1}{2q}$ times the Series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} +$ &c. for the Length of the Whole. But whether falſe or true, I leave to thoſe, who are at Leiſure to try. However all the Methods I have elſewhere ſeen for determining the ſame I am ſure are falſe.

XIII.

Of the Reciprocal Spiral.

IF in the Deſcription of the Spiral, the ſpiral Radius $MC=y$ has always the ſame Proportion to its correſponding conterminant circular Radius $NC=r$, which the whole Revolution cq to A has to $cq+NA$ or $cq+u$; then is the Spiral ſaid to be *Reciprocal*.

Hence we have $y=\frac{cqr}{cq+u}$, $u=cq\times\frac{r-y}{y}$, $y^{\circ}=-\frac{cqru^{\circ}}{\overline{cq+u}^{2}}$, and $u^{\circ}=\frac{-cqry^{\circ}}{yy}$. *Conſect.* 53.

Therefore, for $\frac{yyu^{\circ}}{2r}$ (Theo. 7.) we have $\frac{c^{2}q^{2}ru^{\circ}}{2\times\overline{cq+u}^{2}}=\frac{ru^{\circ}}{2}$ multiplying the Series $1-\frac{2u}{cq}+\frac{3u^{2}}{\overline{cq}^{2}}-\frac{4u^{3}}{\overline{cq}^{3}}+\frac{5u^{4}}{\overline{cq}^{4}}-\frac{6u^{5}}{\overline{cq}^{5}}+$&c. the Subject of which is $\frac{ru}{2}$ multiplying the Series $1-\frac{u}{cq}+\frac{u^{2}}{\overline{cq}^{2}}-\frac{u^{3}}{\overline{cq}^{3}}+\frac{u^{4}}{\overline{cq}^{4}}-\frac{u^{5}}{\overline{cq}^{5}}+$ &c. But the laſt Series is equal to $\frac{cq}{cq+u}$. Therefore every ſuch ſpiral Area MACM is equal to $\frac{ru}{2}\times\frac{cq}{cq+u}$. When therefore u is the ſame with q, as it can never be more for the included Area; the whole conterminant Spiral Area AOMHGFCA; whatever c be, is equal to $\frac{rq}{2}\times\frac{c}{c+1}$. That is, the firſt ſpiral Space is $\frac{1}{2}$, the ſecond $\frac{2}{3}$, the third $\frac{3}{4}$, *&c.* the Area of its conterminant Circle. *Conſ.* 54.

Here alſo for $\frac{yyu^{\circ}}{ry^{\circ}}$ (Theo. 9.) we have always $cq=d=RC$ the Subtangent. *Conſect.* 55.

Laſtly, for the Length of the ſpiral Arch, in the firſt Spiral, inſtead of $\frac{pyu^{o}}{dr}$ (Theo. 10.) we have $\frac{qu^{o}}{\overline{q+u}^{2}}\times\overline{\overline{q+u}^{2}+rr}^{\frac{1}{2}}$, or (putting $q+u=a$, and $r=1$) $\frac{qa^{o}}{aa}\times\overline{aa+1}^{\frac{1}{2}}-q$ multiplying the Series $\frac{a^{o}}{a}+\frac{a^{o}}{2a^{3}}-\frac{a^{o}}{8a^{5}}+\frac{a^{o}}{16a^{7}}-\frac{5a^{o}}{128a^{9}}+\frac{7a^{o}}{256a^{11}}-\frac{21a^{o}}{1024a^{13}}+$ &c. for its Ultimator, whoſe Subject is $o-\frac{q}{4a^{2}}+\frac{q}{32a^{4}}-\frac{q}{90a^{6}}+\frac{5q}{896a^{8}}-\frac{7q}{2560a^{10}}+\frac{7q}{4096a^{12}}-$ &c, falſe. The Learner therefore may try the Ultimator under u at his Leiſure. If he proceed with y, the Ultimator is $\frac{y^{o}}{y}\times\overline{qq+yy}^{\frac{1}{2}}=\frac{qy^{o}}{y}+\frac{yy^{o}}{2q}-\frac{y^{3}y^{o}}{8q^{3}}+\frac{y^{5}y^{o}}{16q^{5}}-\frac{5y^{7}y^{o}}{128q^{7}}+\frac{7y^{9}y^{o}}{256q^{9}}-$ *&c.* whoſe Subject is $o+\frac{y^{2}}{4q}-\frac{y^{4}}{32q^{3}}+\frac{y^{6}}{96q^{5}}-\frac{5y^{8}}{1024q^{7}}+\frac{7y^{10}}{2560q^{9}}-$ &c. again falſe. Nor can I ſee how they mend the Matter, who (contrary to all their own Rules of Fluxions do here make q to be Subject of $\frac{qy^{o}}{y}$. For ſtill it is only what is above q, which muſt determine the Length of the ſpiral Arch correſpondent to the circular Arch u: Unleſs we can be ſo abſurd as to imagine that every ſmalleſt ſpiral Arch is greater than the Circumference q of the whole conterminant Circle.

XIV.

Of the Parabolic, *&c.* Spirals.

WHEN the Spiral is ſo generated, that the complemental Arch $NA=u$ of the conterminant Circle, and the complemental Spiral Radius $NM=r-y$ are always to each other as the Abſciſſa and Semiordinate of a common Parabola, whoſe Parameter is b; that is, when we always have $bu=\overline{r-y}^{2}$, then the Spiral is ſaid to be *Parabolic. Conſect.* 56.

Whence, becauſe $y = r - \overline{bu}^{\frac{1}{2}}$, and $u^{\circ} = \frac{-2ry^{\circ} + 2yy^{\circ}}{b}$; therefore (Theo. 7.) for $\frac{yyu^{\circ}}{2r}$ we have $\frac{1}{2} r u^{\circ} - b^{\frac{1}{2}} u^{\frac{1}{2}} u^{\circ} + \frac{b}{2r} u u^{\circ}$ or $\frac{-ry^{2}y^{\circ} + y^{3}y^{\circ}}{br}$; of which the former Subject $\frac{1}{2} r u - \frac{2}{3} b^{\frac{1}{2}} u^{\frac{3}{2}} + \frac{b}{4r} u^{2}$ gives the Area MCAM, and the latter $-\frac{y^{3}}{3b} + \frac{y^{4}}{4br}$ the negative Value of the Area MHGFCM. *Conſect.* 57.

If therefore we put $\frac{rr}{b} = q$ the Circumference of the conterminant Circle, whoſe Radius is r, whence $b = \frac{rr}{q}$ and $b^{\frac{1}{2}} = \frac{r}{q^{\frac{1}{2}}}$; then, when u becomes equal to q, and conſequently y equal to r, from the former we have $\frac{1}{2} rq - \frac{2}{3} b^{\frac{1}{2}} q^{\frac{3}{2}} + \frac{b}{4r} qq$, and from the latter $\frac{r^{3}}{3b} - \frac{r^{3}}{4b}$, each equal to $\frac{rq}{2} \times \frac{1}{6}$ or to $\frac{1}{6}$ Part of the conterminant Circle for the whole Spiral Area. *Conſect.* 58.

The Method of drawing a Tangent to this Curve, as a Curve of contrary Flexion has been already ſhewn in the eighth Section.

And the ſame may be done from the Hiperbola, Ciſſoid, Conchoid, or any other Aufugient Curve, *ad infinitum*.

XV.

Of the Logarithmic Spiral.

WHEN the Spiral is ſo generated, that every ſpiral Radius CF, CG, CH, at equal Angles from each other, are always in direct Proportion (*i. e.* CF : CG = CG : CH); or which is the ſame, when every ſpiral Radius MC = y has its reſpective Arch ABDEN = a for its reſpective Logarithm; then is ſuch Spiral ſaid to be *Logarithmic*. *Conſect.* 59.

In this Curve put q equal to the Circumference of the Circle, whoſe Radius is $AC = r$; then, becauſe $MC = y$, and $AC = a$, are in continued Proportion, make $r = y^n$ (the Exponent n from the Nature of Logarithms being infinite); whence $q = \frac{qy^n}{r}$. But it has been ſhewn (Sect. VII. Theo. 26.) that the Ultimators of the Logarithms of all Powers of the ſame Number, as y^1 and y^n, are to one another as the Exponents of thoſe Powers, or as 1 to n. Since therefore in this Caſe $\overset{\circ}{a}$ is the Ultimator of a the Logarithm of y, we have $1 : n = \overset{\circ}{a} : n\overset{\circ}{a}$ the Ultimator of the Logarithm of y^n or r; which (becauſe it is here the Logarithm of q or $\frac{qy^n}{r}$ the Circumference of the conterminant Circle) gives $na = \frac{qy^n}{r}$. Whence $a = \frac{qy^n}{nr}$, and $\overset{\circ}{a} = \overset{\circ}{u} = \frac{q}{r} y^{n-1} \overset{\circ}{y}$. But n is infinite, and conſequently as ſuch can bear no Proportion to any finite Quantity, therefore by ſubſtituting r for its equal y^n we have $\overset{\circ}{a} = \overset{\circ}{u} = \frac{q\overset{\circ}{y}}{y}$. *Confect.* 60.

Whence, for $\frac{yy\overset{\circ}{u}}{2r}$ (Theo. 7.) we have $\frac{qy\overset{\circ}{y}}{2r}$ whoſe Subject $\frac{qyy}{4r}$ equals the ſpiral Area. So that when y equals r the whole Area ANHGFCA is $\frac{qr}{4}$, or half the Area of the conterminant Circle. *Confect.* 61.

Alſo, for $\frac{yy\overset{\circ}{u}}{r\overset{\circ}{y}}$ (Theo. 9.) we have $\frac{qy}{r} = RC = d$ the Subtangent to the Point M. *Confect*, 62.

And, if it might be depended upon for $\frac{py\overset{\circ}{u}}{dr}$ (Theo. 10.) we have $\overline{qq+rr}^{\frac{1}{2}} \times \frac{\overset{\circ}{y}}{r}$, whoſe Subject $\overline{qq+rr}^{\frac{1}{2}} \times \frac{y}{r}$ is the Length of the ſpiral Arch MHGFC. Conſequently, when y equals r, the Length of the Whole is $\overline{qq+rr}^{\frac{1}{2}}$. Which, ſince it is more than q the Circumference of the conterminant Circle, can only be underſtood of an infinite Circumvolution about the center C. *Confect.* 63.

The

The Concluſion.

AND thus I conceive I have given a ſufficient Number of Examples to prepare the Learner for his further Progreſs in this Branch of Learning; which, as it is wholly converſant about Quantity, muſt be as infinite as Quantity is in itſelf.

But let it be obſerved once for all, that the Direct Method of Ultimators is that only, which, ſtrictly ſpeaking, can be always admitted as Demonſtrative. And the Inverſe Method can hardly be allowed, as ſuch, any further in the Doctrine of Curves than what relates to Vertical Curves alone, as founded upon the firſt ſix Sections.

To ſhew the Imperfection of the Inverſe Method (beſide what appears above in the Rectification of ſpiral Arches) I need only inſtance in the *54th* Conſectary, where the Area of the Reciprocal Spiral is found equal to $\frac{rq}{2} \times \frac{c}{c+1}$. If this had been ſought by y inſtead of u; then (becauſe $u^{\circ} = \frac{cqry^{\circ}}{yy}$) for $\frac{yyu^{\circ}}{2r}$ we ſhou'd have had $\frac{cqy^{\circ}}{2}$; and conſequently $\frac{rq}{2} \times c$ muſt have been for the Area of the Whole: a Thing impoſſible in itſelf, as making in the firſt Spiral, the Area equal to that of the conterminant Circle; a Part equal to the Whole.

When all therefore is done, notwithſtanding every Pretenſion to the contrary, the demonſtrative Menſuration at leaſt of ſpiral Lines and Surfaces muſt ſtill be left among the Number of Mathematical *Deſiderata*.

FINIS.